Estimating the number of negative eigenvalues of Schrödinger operators

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1 Upper estimate in $\mathbb{R}^n$, $n \geq 3$

1.1 Introduction and statement

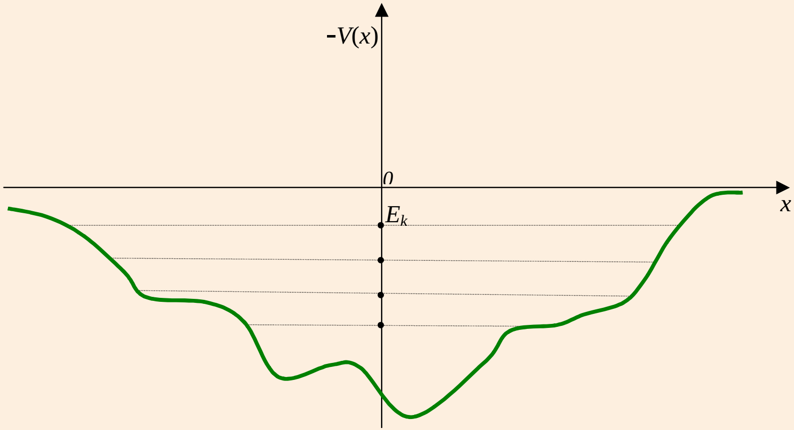
Given a non-negative L^1_{loc} function $V(x)$ on $\mathbb{R}^n$, consider the Schrödinger type operator

$$H_V = -\Delta - V$$

where $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ is the classical Laplace operator. More precisely, H_V is defined as a form sum of $-\Delta$ and $-V$, so that, under certain assumptions about V, the operator H_V is self-adjoint in $L^2(\mathbb{R}^n)$.

Denote by $\operatorname{Neg}(H_V)$ the number of negative eigenvalues of H_V (counted with multiplicity), assuming that its spectrum in $(-\infty, 0)$ is discrete. For example, the latter is the case when $V(x) \to 0$ as $x \to \infty$. We are are interested in obtaining estimates of $\operatorname{Neg}(H_V)$ in terms of the potential V.

Suppose that $-V$ is an attractive potential field in quantum mechanics. Then H_V is the Hamiltonian of a particle that moves in this field, and the negative eigenvalues of H_V correspond to so called *bound states* of the particle, that is, the negative energy levels E_k that are inside a potential well.



Hence, $\mathrm{Neg}\,(H_V)$ determines the number of bound states of the system. In particular, if $-V$ is the potential field of an electron in an atom, then $\mathrm{Neg}\,(H_V)$ is the maximal number of possible electron orbits in the atom.

Estimates of $\mathrm{Neg}\,(H_V)$, especially upper bounds, are of paramount importance for quantum mechanics.

We start with a famous theorem of Cwikel-Lieb-Rozenblum.

Theorem 1.1 *Assume $n \geq 3$ and $V \in L^{n/2}(\mathbb{R}^n)$. Then H_V can be defined as a self-adjoint operator, its negative spectrum is discrete, and the following estimate is true*

$$\operatorname{Neg}(H_V) \leq C_n \int_{\mathbb{R}^n} V(x)^{n/2}\, dx. \tag{1.1}$$

This estimate was proved independently by the above named authors in 1972-1977. Later Lieb used (1.1) to prove the stability of the matter in the framework of quantum mechanics.

The estimate (1.1) implies that, for a large parameter α,

$$\operatorname{Neg}(\alpha V) = O\left(\alpha^{n/2}\right) \quad \text{as } \alpha \to \infty. \tag{1.2}$$

This is a so called semi-classical asymptotic (that corresponds to letting $\hbar \to 0$), and it is expected from another consideration that $\operatorname{Neg}(\alpha V)$ should behave as $\alpha^{n/2}$, at least for a reasonable class of potentials.

1.2 Counting function

Before the proof of Theorem 1.1, let us give an exact definition of the operator H_V and its counting function. Given a potential V in $\mathbb{R}^n$, that is, a non-negative function from $L^1_{loc}(\mathbb{R}^n)$, define the bilinear energy form by

$$\mathcal{E}_V(f,g) = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g dx - \int_{\mathbb{R}^n} V f g dx$$

for all $f, g \in \mathcal{D} := C_0^\infty(\mathbb{R}^n)$, and the corresponding quadratic form $\mathcal{E}_V(f) := \mathcal{E}_V(f,f)$.

For any open set $\Omega \subset \mathbb{R}^n$, we consider a restriction of $\mathcal{E}_V$ to $\mathcal{D}_\Omega := C_0^\infty(\Omega)$. The form $(\mathcal{E}_V, \mathcal{D}_\Omega)$ is called *closable* in $L^2(\Omega)$ if

1. it is semi-bounded below, that is, for some constant $K \geq 0$,

$$\mathcal{E}_V(f) \geq -K \|f\|_2^2 \quad \text{for all } f \in \mathcal{D}_\Omega;$$

2. and, for any sequence $\{f_n\} \subset \mathcal{D}_\Omega$,

$$\|f_n\|_2 \to 0 \text{ and } \mathcal{E}_V(f_n - f_m) \to 0 \implies \mathcal{E}_V(f_n) \to 0.$$

A closable form $(\mathcal{E}_V, \mathcal{D}_\Omega)$ has a unique extension to a subspace $\mathcal{F}_{V,\Omega}$ of $L^2(\Omega)$ so that $\mathcal{F}_{V,\Omega}$ is a Hilbert space with respect to the inner product

$$(f, g)_{\mathcal{E}} := \mathcal{E}_V(f, g) + (K+1)(f, g), \tag{1.3}$$

(that is, $(\mathcal{E}_V, \mathcal{F}_{V,\Omega})$ is *closed*) and $\mathcal{D}_\Omega$ is dense in $\mathcal{F}_{V,\Omega}$.

Being a closed form, $(\mathcal{E}_V, \mathcal{F}_{V,\Omega})$ has the *generator* $H_{V,\Omega}$ that can be defined as an (unbounded) operator in $L^2(\Omega)$ with a maximal possible domain $\operatorname{dom}(H_{V,\Omega}) \subset \mathcal{F}_{V,\Omega}$ such that

$$\mathcal{E}_V(f, g) = (H_{V,\Omega} f, g) \quad \forall f \in \operatorname{dom}(H_{V,\Omega}) \text{ and } g \in \mathcal{F}_{V,\Omega}. \tag{1.4}$$

Then $H_{V,\Omega}$ is a self-adjoint operator in $L^2(\Omega)$.

For example, for $f, g \in \mathcal{D}_\Omega$ we have

$$\mathcal{E}_V(f, g) = \int_\Omega \nabla f \cdot \nabla g dx - \int_\Omega V f g dx = \int_\Omega (-\Delta f - V f) g dx$$

so that

$$H_{V,\Omega} f = -\Delta f - V f.$$

Since the operator $H_{V,\Omega}$ is self-adjoint, the spectrum of $H_{V,\Omega}$ is real and semi-bounded below. The *counting function* $\mathcal{N}_\lambda$ of $H_{V,\Omega}$ is defined by

$$\mathcal{N}_\lambda\left(H_{V,\Omega}\right) = \dim \operatorname{Im} \mathbf{1}_{(-\infty,\lambda)}\left(H_{V,\Omega}\right), \tag{1.5}$$

where $\mathbf{1}_{(-\infty,\lambda)}\left(H_{V,\Omega}\right)$ is the spectral projector of $H_{V,\Omega}$ of the interval $(-\infty,\lambda)$. For example, if the spectrum of $H_{V,\Omega}$ is discrete and $\{\varphi_k\}$ is an orthonormal basis of eigenfunctions with eigenvalues $\{\lambda_k\}$ then $\mathbf{1}_{(-\infty,\lambda)}\left(H_{V,\Omega}\right)$ is the projection on the subspace of $L^2\left(\Omega\right)$ spanned by all φ_k with $\lambda_k < \lambda$. It follows that $\mathcal{N}_\lambda\left(H_{V,\Omega}\right)$ is the number of eigenvalues $\lambda_k < \lambda$ counted with multiplicity. The definition (1.5) has advantage that it always makes sense.

Lemma 1.2 *The following identity is true for all real λ:*

$$\mathcal{N}_\lambda\left(H_{V,\Omega}\right) = \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_\Omega \text{ and } \mathcal{E}_V\left(f\right) < \lambda \left\|f\right\|_2^2 \quad \forall f \in \mathcal{V}\setminus\{0\}\right\}, \tag{1.6}$$

where $\mathcal{V} \prec \mathcal{D}_\Omega$ means that $\mathcal{V}$ is a subspace of $\mathcal{D}_\Omega$. In fact, it suffices to restrict sup *to finite dimensional subspaces $\mathcal{V}$.*

For example, if the spectrum of $H_{V,\Omega}$ is discrete and $\{\varphi_k\}$ is an orthonormal basis of eigenfunctions with eigenvalues $\{\lambda_k\}$ then the condition $\mathcal{E}_V(f) < \lambda \|f\|_2^2$ is satisfied exactly for $f = \varphi_k$ provided $\lambda_k < \lambda$, because

$$\mathcal{E}_V(\varphi_k) = (H_{V,\Omega}\varphi_k, \varphi_k) = \lambda_k(\varphi_k, \varphi_k) < \lambda \|\varphi_k\|_2^2 .$$

The optimal space $\mathcal{V}$ in (1.7) is spanned by all $\{\varphi_k\}$ with $\lambda_k < \lambda$, and its dimension is equal to $\mathcal{N}_\lambda(H_{V,\Omega})$.

There is also a version of counting function with non-strict inequality:

$$\mathcal{N}_\lambda^*(H_{V,\Omega}) = \dim \operatorname{Im} \mathbf{1}_{(-\infty,\lambda]}(H_{V,\Omega}) .$$

Then the following identity is true:

$$\mathcal{N}_\lambda^*(H_{V,\Omega}) = \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{F}_{V,\Omega} \text{ and } \mathcal{E}_V[f] \le \lambda\mu[f] \quad \forall f \in \mathcal{V}\right\}. \tag{1.7}$$

1.3 Reduction to operator $\frac{1}{V}\Delta$

For the sake of proof of Theorem 1.1, we will assume that $V > 0$ and, moreover, $\frac{1}{V} \in L^1_{loc}(\mathbb{R}^n)$. Then by approximation argument one can handle a general case. Set $H_V \equiv H_{V,\mathbb{R}^n}$. Our aim is to prove the upper bound

$$\mathcal{N}_0(H_V) \leq C_n \int_{\mathbb{R}^n} V^{n/2} dx$$

for the number $\mathcal{N}_0(H_V)$ of negative eigenvalues. By an approximation argument the same estimate will hold for the number $\mathcal{N}_0^*(H_V)$ of non-positive eigenvalues.

For $\lambda = 0$ the identity (1.6) becomes

$$\mathcal{N}_0(H_{V,\Omega}) = \sup\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_\Omega \text{ and } \mathcal{E}_V(f) < 0\ \forall f \in \mathcal{V} \setminus \{0\}\}. \quad (1.8)$$

The condition $\mathcal{E}_V(f) < 0$ here is equivalent to

$$\int_\Omega |\nabla f|^2 dx - \int_\Omega V f^2 dx < 0 \quad (1.9)$$

for all non-zero $f \in \mathcal{V}$ where $\mathcal{V}$ is a subspace of $\mathcal{D}_\Omega$.

We will interpret this inequality in terms of the counting function of another operator. Consider a new measure μ defined by

$$d\mu = V(x)\, dx$$

and the energy form

$$\mathcal{E}(f) = \int_{\mathbb{R}^n} |\nabla f|^2\, dx$$

for $f \in \mathcal{D}_\Omega$. Then (1.9) can be rewritten in the form $\mathcal{E}(f) < \|f\|_{2,\mu}^2$ so that

$$\mathcal{N}_0(H_{V,\Omega}) = \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_\Omega \text{ and } \mathcal{E}(f) < \|f\|_{2,\mu}^2 \ \forall f \in \mathcal{V} \setminus \{0\}\right\}. \tag{1.10}$$

The right hand side here is the counting function of another operator. Indeed, denoted by $\mathcal{L}_{V,\Omega}$ the generator of the energy form $(\mathcal{E}, \mathcal{D}_\Omega)$ in $L^2(\Omega, \mu)$. This form can be shown to be closable, so that its generator $\mathcal{L}_{V,\Omega}$ is a self-adjoint operator in $L^2(\Omega, \mu)$. Note also that this operator is positive definite because so is $\mathcal{E}$.

By definition, we have, for all $f, g \in \operatorname{dom}(\mathcal{L}_{V,\Omega})$,

$$\mathcal{E}(f, g) = (\mathcal{L}_{V,\Omega} f, g)_\mu .$$

In particular, for $f, g \in \mathcal{D}_\Omega$ this implies

$$-\int_\Omega (\Delta f)\, g dx = \int_\Omega \nabla f \cdot \nabla g \; dx = \int_\Omega (\mathcal{L}_{V,\Omega} f)\, gV dx,$$

whence $\mathcal{L}_{V,\Omega} f = -\frac{1}{V}\Delta f$ that is, $\mathcal{L}_{V,\Omega} = -\frac{1}{V}\Delta$.

The counting function $\mathcal{N}_\lambda(\mathcal{L}_{V,\Omega})$ of the operator $\mathcal{L}_{V,\Omega}$ is defined exactly as for $H_{V,\Omega}$. Lemma 1.2 for this operator means that

$$\mathcal{N}_\lambda(\mathcal{L}_{V,\Omega}) = \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_\Omega \text{ and } \mathcal{E}(f) < \lambda \|f\|_{2,\mu}^2 \quad \forall f \in \mathcal{V} \setminus \{0\}\right\}. \tag{1.11}$$

For $\lambda = 1$ the right hand side of (1.11) coincides with that of (1.10), which implies

$$\mathcal{N}_0(H_{V,\Omega}) = \mathcal{N}_1(\mathcal{L}_{V,\Omega}). \tag{1.12}$$

In particular, for the case $\Omega = \mathbb{R}^n$, we have $\mathcal{N}_0(H_V) = \mathcal{N}_1(\mathcal{L}_V)$. The identity (1.12) is called *Birman-Schwinger principle.*

Informally the identity (1.12) reflects the equivalence of the inequalities $-\Delta - V \leq 0$ and $-\frac{1}{V}\Delta \leq 1$ that are understood in the sense of quadratic forms.

1.4 Case of small V

Here we illustrate the usage of (1.12) by proving a particular case of Theorem 1.1 as follows.

Proposition 1.3 *If $n \geq 3$ then there is a constant $c_n > 0$ such that*

$$\int_{\mathbb{R}^n} V^{n/2} dx < c_n \Rightarrow \mathcal{N}_0 \left(H_V \right) = 0.$$

Proof. By (1.12) we need to prove that the spectrum of $\mathcal{L}_V$ below 1 is empty, that is,

$$\inf \operatorname{spec} \mathcal{L}_V \geq 1.$$

This is equivalent to the claim that the operator $\mathcal{L}_V$ in $L^2 \left(\mathbb{R}^n, \mu \right)$ is invertible and

$$\left\| \mathcal{L}_V^{-1} \right\| \leq 1.$$

The inverse operator is defined by

$$\mathcal{L}_V^{-1} f = u \quad \Leftrightarrow \quad \mathcal{L}_V u = f,$$

where $f \in L^2(\mathbb{R}^n, \mu)$ and $u \in \operatorname{dom}(\mathcal{L}_V)$. Hence, it suffices to prove that

$$\mathcal{L}_V u = f \quad \Rightarrow \|u\|_{2,\mu} \leq \|f\|_{2,\mu}.$$

Multiplying $\mathcal{L}_V u = f$ by u and integrating against μ, we obtain

$$\mathcal{E}(u) = (\mathcal{L}_V u, u)_\mu = (f, u)_\mu$$

that is,

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx = \int_{\mathbb{R}^n} u f d\mu.$$

By Sobolev inequality, we have

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq c_n \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}}.$$

Note that this is the only place where $n > 2$ is used.

Using the Hölder inequality and the above lines, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} u^2 V\,dx &\le \left(\int_{\mathbb{R}^n} |u|^{2\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \left(\int_{\mathbb{R}^n} V^{\frac{n}{2}}\,dx\right)^{\frac{2}{n}} \\
&\le c_n^{-1}\left(\int_{\mathbb{R}^n} |\nabla u|^2\,dx\right)\left(\int_{\mathbb{R}^n} V^{\frac{n}{2}}\,dx\right)^{\frac{2}{n}} \qquad (1.13) \\
&= c_n^{-1}\left(\int_{\mathbb{R}^n} uf\,d\mu\right)\left(\int_{\mathbb{R}^n} V^{\frac{n}{2}}\,dx\right)^{\frac{2}{n}} \\
&\le c_n^{-1}\left(\int_{\mathbb{R}^n} f^2 d\mu\right)^{1/2}\left(\int_{\mathbb{R}^n} u^2 d\mu\right)^{1/2}\left(\int_{\mathbb{R}^n} V^{\frac{n}{2}}\,dx\right)^{\frac{2}{n}}
\end{aligned}$$

whence

$$\|u\|_{2,\mu} \le c_n^{-1}\left(\int_{\mathbb{R}^n} V^{\frac{n}{2}}\,dx\right)^{\frac{2}{n}}\|f\|_{2,\mu}\,.$$

Clearly, if $\int_{\mathbb{R}^n} V^{\frac{n}{2}}\,dx$ small enough then $\|u\|_{2,\mu} \le \|f\|_{2,\mu}$, which was to be proved. ■

The argument in the proof of Proposition 1.3 allows to prove another part of Theorem 1.1.

Proposition 1.4 *If $V \in L^{n/2}(\mathbb{R}^n)$ then the form $(\mathcal{E}_V, \mathcal{D})$ is closable. Consequently, the operator H_V is defined as a self-adjoint operator in $L^2(\mathbb{R}^n)$.*

Proof. It follows from the hypothesis that, for any $\varepsilon > 0$, V can be split to a sum of two potentials $V = V_1 + V_2$ where

$$\|V_1\|_{n/2} \le \varepsilon \text{ and } V_2 \in L^\infty.$$

It follows from (1.13) that

$$\mathcal{E}(u) \ge c_n \left(\int_{\mathbb{R}^n} V_1^{n/2} dx \right)^{-2/n} \int_{\mathbb{R}^n} u^2 V_1 dx \ge c_n \varepsilon^{-1} \int_{\mathbb{R}^n} u^2 V_1 dx.$$

Choosing ε sufficiently small, we obtain $c_n \varepsilon^{-1} \ge 2$ whence

$$\begin{aligned} \int_{\mathbb{R}^n} u^2 V dx &= \int_{\mathbb{R}^n} u^2 V_1 dx + \int_{\mathbb{R}^n} u^2 V_2 dx \\ &\le \frac{1}{2}\mathcal{E}(u) + K \|u\|_2^2, \end{aligned} \quad (1.14)$$

where $K = \|V_2\|_{L^\infty}$. In particular, we see that

$$\mathcal{E}_V(u) = \mathcal{E}(u) - \int_{\mathbb{R}^n} u^2 V dx \geq -K \|u\|_2^2$$

so that the form $\mathcal{E}_V$ is semi-bounded below. By a standard result from the theory of quadratic forms, (1.14) implies that the form $\mathcal{E}_V$ is closed in the domain $W^{1,2}(\mathbb{R}^n)$, which finishes the proof. ■

1.5 Proof of Theorem 1.1 in general case

The proof below is due to Li and Yau '83 but it is presented here from somewhat different angle.

In a precompact domain Ω the operator $\mathcal{L}_{V,\Omega}$ has discrete positive spectrum. Denote its eigenvalues by $\lambda_k(\Omega)$, where $k = 1, 2, ...,$ so that the sequence $\{\lambda_k(\Omega)\}$ is increasing, and each eigenvalue is counted with multiplicity. The main part of the proof of Theorem 1.1 is contained in the following statement.

Theorem 1.5 (AG, Yau 2003) *Assume that there is a Radon measure ν in $\mathbb{R}^n$ and $\alpha > 0$ such that, for all precompact open sets Ω,*

$$\lambda_1(\Omega) \geq \nu(\Omega)^{-\alpha}. \tag{1.15}$$

Then, for any positive integer k and any precompact open set Ω,

$$\lambda_k(\Omega) \geq c\left(\frac{k}{\nu(\Omega)}\right)^{\alpha}, \tag{1.16}$$

where $c = c(\alpha) > 0$.

For example, if $V=1$ then $\mathcal{L}_{V,\Omega}$ is the Laplace operator $-\Delta$ with the Dirichlet boundary condition on $\partial\Omega$. The hypothesis (1.15) is satisfies if ν is a multiple of the Lebesgue measure as by the Faber-Krahn inequality

$$\lambda_1(\Omega) \geq c_n (\operatorname{vol}\Omega)^{-2/n}.$$

Then (1.16) becomes

$$\lambda_k(\Omega) \geq c_n' \left(\frac{k}{\operatorname{vol}\Omega}\right)^{2/n},$$

that is also known to be true. Moreover, it matches the Weyl's asymptotic formula $\lambda_k(\Omega) \sim \widetilde{c}_n \left(\frac{k}{\operatorname{vol}\Omega}\right)^{2/n}$ as $k \to \infty$.

The point of Theorem 1.5 is that V in the definition of $\mathcal{L}_{V,\Omega}$ can be arbitrary and measure ν can be arbitrary. By the way, there is no restriction of the dimension n in Theorem 1.5. Moreover, exactly in this form it is true on any Riemannian manifold instead of $\mathbb{R}^n$.

Let us show how Theorem 1.5 implies Theorem 1.1. Let us use the variational principle:

$$\lambda_1(\Omega) = \inf_{u\in\mathcal{D}_\Omega} \frac{(\mathcal{L}_{V,\Omega}u, u)_\mu}{(u,u)_\mu} = \inf_{u\in\mathcal{D}_\Omega} \frac{\mathcal{E}(u)}{(u,u)_\mu}.$$

Using again the Sobolev inequality

$$\int_{\Omega} |\nabla u|^2 \, dx \geq c_n \left(\int_{\Omega} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}}$$

and the Hölder inequality

$$(u, u)_{\mu} = \int_{\Omega} u^2 V dx \leq \left(\int_{\Omega} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \left(\int_{\Omega} V^{n/2} dx \right)^{\frac{2}{n}},$$

we obtain

$$\frac{\mathcal{E}(u)}{(u, u)_{\mu}} \geq c_n \left(\int_{\Omega} V^{n/2} dx \right)^{-\frac{2}{n}}.$$

Hence, setting $d\nu = c_n^{-n/2} V^{n/2} dx$ and minimizing in u, we obtain

$$\lambda_1(\Omega) \geq \nu(\Omega)^{-2/n}.$$

By Theorem 1.5, we conclude that

$$\lambda_k(\Omega) \geq c \left(\frac{k}{\nu(\Omega)} \right)^{2/n}. \tag{1.17}$$

We need to estimate the counting function

$$\mathcal{N}_1 (\mathcal{L}_{V,\Omega}) = \# \{k : \lambda_k (\Omega) < 1\} .$$

By (1.17), $\lambda_k (\Omega) < 1$ implies $k \le C\nu (\Omega)$ whence also

$$\mathcal{N}_1 (\mathcal{L}_{V,\Omega}) \le C\nu (\Omega) = C \int_\Omega V^{n/2} dx.$$

It follows by (1.12) that also

$$\mathcal{N}_0 (H_{V,\Omega}) \le C \int_\Omega V^{n/2} dx \le C \int_{\mathbb{R}^n} V^{n/2} dx. \tag{1.18}$$

We are left to pass from $H_{V,\Omega}$ to $H_{V,\mathbb{R}^n}$. Recall that

$$\mathcal{N}_0 (H_{V,\mathbb{R}^n}) = \sup \{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_{\mathbb{R}^n}, \quad \mathcal{E}_V (f) < 0 \; \forall f \in \mathcal{V} \setminus \{0\}\} ,$$

where $\mathcal{V}$ is a finite-dimensional subspace of $\mathcal{D}_{\mathbb{R}^n}$. For any such $\mathcal{V}$ there exists a precompact open set Ω containing $\operatorname{supp} f$ for all $f \in \mathcal{V}$ (for it suffices to have $\operatorname{supp} f \subset \mathcal{V}$ for the elements of a basis of $\mathcal{V}$). Hence, $\mathcal{V} \prec \mathcal{D}_\Omega$ and by (1.18) $\dim \mathcal{V} \le C \int_{\mathbb{R}^n} V^{n/2} dx$, whence the same estimate for $\mathcal{N}_0 (H_{V,\mathbb{R}^n})$ follows. ■

Brief summary

We prove the following theorem.

Theorem 1.1. *If V is a non-negative potential in $\mathbb{R}^n$ with $n \geq 3$ then for the operator $H_V = -\Delta - V$,*

$$\mathcal{N}_0 (H_V) \leq C_n \int_{\mathbb{R}^n} V(x)^{n/2} \, dx. \tag{1.1}$$

This was reduced to the following theorem.

Theorem 1.5. *For any bounded domain $\Omega \subset \mathbb{R}^n$, denote by $\lambda_k(\Omega)$ the k-th eigenvalue of the operator $\mathcal{L}_{V,\Omega} = -\frac{1}{V}\Delta$ (with the Dirichlet boundary condition on $\partial\Omega$). Assume that there is a Radon measure ν in $\mathbb{R}^n$ and $\alpha > 0$ such that, for all bounded domains Ω,*

$$\lambda_1(\Omega) \geq \nu(\Omega)^{-\alpha}. \tag{1.15}$$

Then, for any positive integer k and any precompact open set Ω,

$$\lambda_k(\Omega) \geq c \left(\frac{k}{\nu(\Omega)} \right)^{\alpha}, \tag{1.16}$$

where $c = c(\alpha) > 0$.

1.6 Nash inequality

For the proof of Theorem 1.5 we need a Nash type inequality.

Lemma 1.6 *Assume that* (1.15) *holds, that is, for all precompact open sets* Ω,

$$\lambda_1(\Omega) \geq \nu(\Omega)^{-\alpha}.$$

Then, for all such Ω *and non-negative* $f \in \mathcal{D}_\Omega$,

$$\mathcal{E}(f) \geq c \left(\int_\Omega f^2 d\mu \right)^{1+\alpha} \left(\int_\Omega f d\mu \int_\Omega f d\nu \right)^{-\alpha}, \tag{1.19}$$

where $c = 2^{-2\alpha-1}$.

Remark. If $V \equiv 1$ then both μ and ν are Lebesgue measures, $\alpha = 2/n$, and (1.19) becomes

$$\mathcal{E}(f) \geq \|f\|_2^{2+4/n} \|f\|_1^{-4/n},$$

which is a classical Nash inequality.

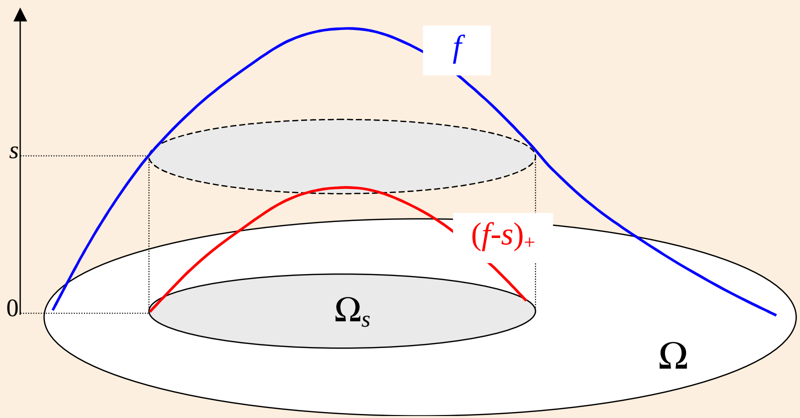
Proof. Fix $s > 0$ and observe that

$$\mathcal{E}\left((f-s)_+\right) \leq \mathcal{E}\left(f\right). \tag{1.20}$$

Set

$$\Omega_s := \{x \in \Omega : f(x) > s\}$$

and note that supp $(f-s)_+ \subset \overline{\Omega}_s \subset \Omega$.



It follows from the variational property of $\lambda_1(\Omega_s)$ and from (1.20), that

$$\int_\Omega (f-s)_+^2\,d\mu = \int_{\Omega_s} (f-s)_+^2\,d\mu \leq \frac{\mathcal{E}\left((f-s)_+\right)}{\lambda_1(\Omega_s)} \leq \frac{\mathcal{E}\,(f)}{\lambda_1(\Omega_s)}. \tag{1.21}$$

Since

$$\nu\left(\Omega_s\right) \leq \frac{1}{s}\int_\Omega f d\nu$$

we obtain by hypothesis

$$\frac{1}{\lambda_1\left(\Omega_s\right)} \leq \nu\left(\Omega_s\right)^\alpha \leq s^{-\alpha}\left(\int_\Omega f d\nu\right)^a.$$

Substituting into (1.21) and using

$$f^2 - 2sf \leq (f-s)_+^2\,,$$

we obtain

$$\int_\Omega f^2 d\mu - 2s\int_\Omega f d\mu \leq s^{-\alpha}\left(\int_\Omega f d\nu\right)^\alpha \mathcal{E}\,(f)\,. \tag{1.22}$$

Let us choose s from the condition

$$2s\int_{\Omega} f\,d\mu = \frac{1}{2}\int_{\Omega} f^2 d\mu.$$

With this s we obtain

$$\frac{1}{2}\int_{\Omega} f^2 d\mu \leq \left(\frac{1}{4}\frac{\int_{\Omega} f^2 d\mu}{\int_{\Omega} f\,d\mu}\right)^{-\alpha}\left(\int_{\Omega} f\,d\nu\right)^{\alpha}\mathcal{E}\left(f\right)$$

whence

$$\left(\int_{\Omega} f^2 d\mu\right)^{1+\alpha} \leq 2^{2\alpha+1}\left(\int_{\Omega} f\,d\mu\right)^{\alpha}\left(\int_{\Omega} f\,d\nu\right)^{\alpha}\mathcal{E}\left(f\right),$$

and (1.19) follows. ■

1.7 Proof of Theorem 1.5

In the proof we work with the heat semigroup $\{P_t\}_{t\geq 0}$ of the operator $\mathcal{L}_{V,\Omega}$, that is defined by

$$P_t^{\Omega} = e^{-t\mathcal{L}_{V,\Omega}}.$$

Since $\mathcal{L}_{V,\Omega}$ is a self-adjoint non-negative definite operator in $L^2(\Omega,\mu)$, the operator P_t^{Ω} is bounded self-adjoint operator in $L^2(\Omega,\mu)$ for any $t\geq 0$. In fact, it is an integral operator:

$$P_t^{\Omega} f(x) = \int_{\Omega} p_t^{\Omega}(x,y)\, f(y)\, d\mu(y)$$

where $p_t^{\Omega}(x,y)$ is the *heat kernel* of $\mathcal{L}_{V,\Omega}$. We will use the following general properties of the heat kernel:

1. positivity: $p_t(x,y) \geq 0$;

2. the symmetry: $p_t^{\Omega}(x,y) = p_t^{\Omega}(y,x)$;

3. the semigroup identity

$$\int_{\Omega} p_t^{\Omega}(x,z)\, p_s^{\Omega}(z,y)\, d\mu(z) = p_{t+s}^{\Omega}(x,y);$$

4. the total mass inequality:

$$\int_{\Omega} p_t^{\Omega}(x,y)\, d\mu(y) \leq 1.$$

The last step before the proof of Theorem 1.5 is the following lemma.

Lemma 1.7 *If* (1.15) *holds, that is,* $\lambda_1(\Omega) \geq \nu(\Omega)^{-\alpha}$, *then, for any precompact open set* Ω,

$$\int_{\Omega} p_t^{\Omega}(x,x)\, d\mu(x) \leq \frac{C\nu(\Omega)}{t^{1/\alpha}}. \tag{1.23}$$

where $C = C(\alpha)$.

Proof. Fix $s > 0$, $x \in \Omega$ and consider a function

$$f = p_s^{\Omega}(x, \cdot)$$

and set $u_t = P_t^{\Omega} f$, that is,

$$u_t(y) = \int_{\Omega} p_t^{\Omega}(y,z) f(z)\, d\mu(z) = p_{t+s}^{\Omega}(x,y).$$

Then we have

$$\int_\Omega u_t^2 d\mu = \int_\Omega p_{t+s}^\Omega(x,y) p_{t+s}^\Omega(y,x)\, d\mu(y) = p_{2(t+s)}^\Omega(x,x)\,.$$

On the other hand, by the Nash inequality we have

$$\int_\Omega u_t^2 d\mu \le \left(\int_\Omega u_t d\mu \int_\Omega u_t d\nu\right)^{\frac{\alpha}{\alpha+1}} \left[C\mathcal{E}\left(u_t\right)\right]^{\frac{1}{\alpha+1}}.$$

Using

$$\int_\Omega u_t d\mu = \int_\Omega p_{t+s}^\Omega(x,y)\, d\mu(y) \le 1, \tag{1.24}$$

and

$$\mathcal{E}\left(u_t\right) = \left(\mathcal{L}_{V,\Omega} u_t, u_t\right)_\mu = -\left(\frac{d}{dt}u_t, u_t\right)_\mu = -\frac{1}{2}\frac{d}{dt}\left(u_t, u_t\right)_\mu$$

we obtain

$$\int_\Omega u_t^2 d\mu \le \left(\int_\Omega u_t d\nu\right)^{\frac{\alpha}{\alpha+1}} \left[-\frac{C}{2}\frac{d}{dt}\int_\Omega u_t^2 d\mu\right]^{\frac{1}{\alpha+1}}.$$

Recall that u_t depends in fact on x. Setting

$$v_t\left(x\right) := \int_\Omega u_t^2 d\mu = p_{2(t+s)}^{\Omega}\left(x, x\right),$$

rewrite the previous inequality in the form

$$v_t(x) \le \left(\int_\Omega u_t d\nu\right)^{\frac{\alpha}{\alpha+1}} \left[-\frac{C}{2}\frac{\partial v_t}{\partial t}\right]^{\frac{1}{\alpha+1}}. \tag{1.25}$$

Integrating (1.25) against $d\mu(x)$ and using the Hölder inequality

$$\int F^{\frac{\alpha}{\alpha+1}} G^{\frac{1}{\alpha+1}} d\mu \le \left[\int F d\mu\right]^{\frac{\alpha}{\alpha+1}} \left[\int G d\mu\right]^{\frac{1}{\alpha+1}},$$

we obtain

$$\begin{aligned} \int_\Omega v_t\left(x\right) d\mu\left(x\right) &\le \int \underbrace{\left[\int u_t d\nu\right]}_{F}^{\frac{\alpha}{\alpha+1}} \underbrace{\left[-\frac{C}{2}\frac{\partial v_t}{\partial t}\right]}_{G}^{\frac{1}{\alpha+1}} d\mu(x) \\ &\le \left[\int\int u_t d\nu d\mu(x)\right]^{\frac{\alpha}{\alpha+1}} \left[-\frac{C}{2}\int \frac{\partial v_t}{\partial t} d\mu(x)\right]^{\frac{1}{\alpha+1}}. \end{aligned}$$

Observe that (1.24) implies

$$\int\int u_t(x,\cdot)d\nu d\mu(x) = \int\left(\int u_t(x,\cdot)d\mu(x)\right)d\nu \leq \int_\Omega d\nu = \nu(\Omega). \tag{1.26}$$

Denoting

$$w\left(t\right) := \int_\Omega v_t\left(x\right)d\mu\left(x\right) = \int_\Omega p^\Omega_{2(t+s)}\left(x,x\right)\mu\left(x\right),$$

we obtain from above

$$w(t) \leq \nu(\Omega)^{\frac{\alpha}{\alpha+1}}\left(-\frac{C}{2}\frac{dw}{dt}\right)^{\frac{1}{\alpha+1}}. \tag{1.27}$$

Solving this differential inequality by separation of variables, we obtain

$$w(t) \leq \frac{C'\nu\left(\Omega\right)}{t^{1/\alpha}}.$$

Finally, choosing $s=t$ we obtain $\int_\Omega p^\Omega_{4t}\left(x,x\right)\mu\left(x\right) \leq \frac{C'\nu\left(\Omega\right)}{t^{1/\alpha}}$, which was to be proved. ■

Proof of Theorem 1.5. We need to show that

$$\lambda_k(\Omega) \geq c\left(\frac{k}{\nu(\Omega)}\right)^{\alpha}.$$

Note that

$$\int_{\Omega} p_t^{\Omega}(x,x)\, d\mu(x) = \operatorname{trace} P_t^{\Omega}.$$

On the other hand, all the eigenvalues of P_t^{Ω} are equal to $e^{-t\lambda_k(\Omega)}$, whence

$$\operatorname{trace} P_t^{\Omega} = \sum_{k=1}^{\infty} e^{-t\lambda_k(\Omega)}.$$

Hence, applying (1.23), we obtain

$$\sum_{k=1}^{\infty} e^{-t\lambda_k(\Omega)} \leq \frac{C\nu(\Omega)}{t^{1/\alpha}}.$$

Restricting the summation to the first k terms, we obtain

$$ke^{-t\lambda_k(\Omega)} \leq \frac{C\nu(\Omega)}{t^{1/\alpha}}$$

whence

$$\lambda_k(\Omega) \geq \frac{1}{t} \ln \frac{kt^{1/\alpha}}{C\nu(\Omega)}.$$

Choosing t from the condition

$$\frac{kt^{1/\alpha}}{C\nu(\Omega)} = e,$$

that is,

$$t = \left(Ce\frac{\nu(\Omega)}{k}\right)^{\alpha},$$

we obtain

$$\lambda_k(\Omega) \geq \frac{1}{t} = \left(\frac{1}{Ce}\frac{k}{\nu(\Omega)}\right)^{\alpha},$$

which finishes the proof of Theorem 1.5. ■

1.8 Minimal surfaces

Let M be a two-dimensional manifold immersed in $\mathbb{R}^3$ as an oriented minimal surface. The Riemannian metric on M is induced by the Euclidean structure of $\mathbb{R}^3$. Denote by α the Riemannian area on M.

For any function $f \in C_0^\infty(M)$ and a real parameter ε, consider a deformation of M given by the mapping $x \mapsto x+\varepsilon f(x)\nu(x)$ where $\nu(x)$ is the unit normal vector field on M compatible with the orientation. Since M is a minimal surface, the first variation $\delta\alpha(f)$ of the area functional vanishes. For the second variation, the following formula is known:

$$\delta^2\alpha(f) = \int_M (|\nabla f|^2 + 2Kf^2)d\alpha\,, \tag{1.28}$$

where $K = K(x)$ is the Gauss curvature of M at the point $x \in M$ (since M is minimal, $K(x) \le 0$). If $\delta^2\alpha(f) \ge 0$ for all f then the minimal surface M is called *stable*. In particular, all area minimizers are stable.

However, in general a minimal surface is not necessarily stable. By definition, the *stability index* $\operatorname{ind}(M)$ is the maximal dimension of a linear subspace $\mathcal{V}$ of $C_0^\infty(M)$ such that $\delta^2\alpha(f) < 0$ for any $f \in \mathcal{V} \setminus \{0\}$.

In other words,

$$\operatorname{ind}(M) = \mathcal{N}_0\left(H_V\right)$$

where $H_V = -\Delta + 2K$ and Δ is the Laplace-Beltrami operator on M.

It turns out that for this specific potential $V = -2K$ the upper bound of Theorem 1.1 is satisfied.

Theorem 1.8 (AG, Yau 2003) *For any immersed oriented minimal surface M in $\mathbb{R}^3$, we have*

$$\operatorname{ind}(M) \leq C \int_M |K|\, d\alpha, \tag{1.29}$$

where C is an absolute constant.

The proof goes in the same way as the one of Theorem 1.1 using Theorem 1.5. Using specific properties of Gauss curvature, we prove for the operator $\mathcal{L}_{V,\Omega} = -\frac{1}{V}\Delta$ in $\Omega \subset M$ the eigenvalue estimate

$$\lambda_1\left(\Omega\right) \geq c\mu\left(\Omega\right)^{-1},$$

where $d\mu = |K|\, d\alpha$. By Theorem 1.5 this implies

$$\lambda_k(\Omega) \geq c' \frac{k}{\mu(\Omega)}$$

and then as in the proof of Theorem 1.1,

$$\mathcal{N}_0(H_V) \leq C\mu(M)$$

that is (1.29).

2 Lower estimates in $\mathbb{R}^2$

Here we estimate $\mathcal{N}_0(H_V)$ in $\mathbb{R}^2$.

2.1 A counterexample to the upper bound

In the case $n = 2$, the estimate (1.1) of Theorem 1.1 becomes

$$\mathcal{N}_0(H_V) \leq C \int_{\mathbb{R}^2} V(x)\, dx,$$

which however is *wrong*. To see that, consider in $\mathbb{R}^2$ the potential

$$V(x) = \frac{1}{|x|^2 \ln^2 |x|} \quad \text{if } |x| > e$$

and $V(x) = 0$ if $|x| \leq e$. For this V we have

$$\int_{\mathbb{R}^2} V(x)\, dx < \infty,$$

whereas $\operatorname{Neg}(H_V) = \infty$. Indeed, consider the function

$$f(x) = \sqrt{\ln|x|} \sin\left(\frac{1}{2} \ln\ln|x|\right)$$

that satisfies in the region $\{|x| > e\}$ the differential equation

$$\Delta f + \frac{1}{2} V(x) f = 0.$$

For any positive integer k, function f has constant sign in the ring

$$\Omega_k := \left\{ x \in \mathbb{R}^2 : \pi k < \frac{1}{2} \ln\ln|x| < \pi(k+1) \right\},$$

and vanishes on $\partial\Omega_k$. For each function $f_k = f \mathbf{1}_{\Omega_k}$ we have

$$\begin{aligned}
\mathcal{E}_V(f_k) &= \int_{\Omega_k} |\nabla f_k|^2 \, dx - \int_{\Omega_k} V f_k^2 dx \\
&= -\int_{\Omega_k} f_k \Delta f_k dx - \int_{\Omega_k} V f_k^2 dx \\
&= -\frac{1}{2} \int_{\Omega_k} V f_k^2 dx < 0.
\end{aligned}$$

The same inequality holds for linear combination of functions f_k since the intersection of their supports has measure 0.

Hence, the space $\mathcal{V} = \operatorname{span}\{f_k\}$ has infinite dimension and $\mathcal{E}_V(f) < 0$ for all non-zero $f \in \mathcal{V}$, which implies $\mathcal{N}_0(H_V) = \infty$.

In fact, one can show that no upper bound of the form

$$\mathcal{N}_0(H_V) \leq \int_{\mathbb{R}^2} V(x) W(x) \, dx$$

can be true, no matter how we choose a weight $W(x)$.

2.2 Lower bound of $\mathcal{N}_0(H_V)$

It turns out that in the case $n = 2$, instead of an upper bound, a lower bound in (1.1) is true.

Theorem 2.1 (AG, Netrusov, Yau, 2004) *For any non-negative potential V in $\mathbb{R}^2$,*

$$\mathcal{N}_0(H_V) \geq c \int_{\mathbb{R}^2} V(x)\, dx \tag{2.1}$$

with some absolute constant $c > 0$.

Let us describe an approach to the proof. Since

$$\mathcal{N}_0(H_V) = \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_{\mathbb{R}^2} \text{ and } \mathcal{E}_V(f) < 0\, \forall f \in \mathcal{V} \setminus \{0\}\right\},$$

it suffices to construct a subspace $\mathcal{V}$ of $\mathcal{D}_{\mathbb{R}^2}$ such that $\mathcal{E}_V$ is negative on $\mathcal{V}$ and

$$\dim \mathcal{V} \geq c \int_{\mathbb{R}^2} V(x)\, dx.$$

We will construct $\mathcal{V}$ as span $\{f_k\}$ where $\{f_k\}_{k=1}^N$ is a sequence of functions with disjoint compact supports such that $\mathcal{E}_V(f_k) < 0$. Then $\mathcal{E}_V(f) <$

0 will be true for any non-zero function $f \in \operatorname{span}\{f_k\}$, and $\dim \mathcal{V} = N$. Hence, it suffices to construct a sequence $\{f_k\}_{k=1}^{N}$ of functions with compact disjoint supports such that, for any $k = 1, ..., N$,

$$\int_{\mathbb{R}^2} |\nabla f_k|^2 \, dx < \int_{\mathbb{R}^2} V f_k^2 dx,$$

and

$$N \geq c \int_{\mathbb{R}^2} V(x) \, dx.$$

Each function f_k will be constructed as follows. Fix two reals $0 < r < R$ and consider the annulus

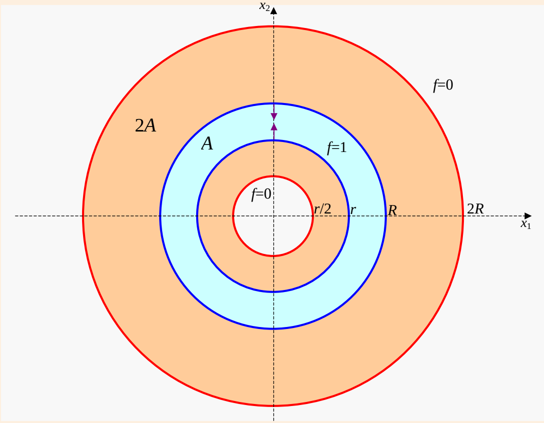
$$A = \left\{ x \in \mathbb{R}^2 : r < |x| < R \right\}$$

and denote by $2A$ the annulus

$$2A = \left\{ x \in \mathbb{R}^2 : \frac{1}{2} r < |x| < 2R \right\}.$$

Consider the following function

$$f(x)=\begin{cases}1, & x\in A,\\ 0, & x\notin 2A,\\ \frac{1}{\ln 2}\ln\frac{2|x|}{r}, & \frac{r}{2}\le|x|\le r,\\ \frac{1}{\ln 2}\ln\frac{2R}{|x|}, & R\le|x|\le 2R.\end{cases}$$



This function f is harmonic in each of the four domains, whence we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla f|^2 \, dx &= \int_{\left\{\frac{r}{2} \leq |x| \leq r\right\}} |\nabla f|^2 \, dx + \int_{\{R \leq |x| \leq 2R\}} |\nabla f|^2 \, dx \\
&= \int_{\partial\left\{\frac{r}{2} \leq |x| \leq r\right\}} f \frac{\partial f}{\partial \nu} dl + \int_{\partial\{R \leq |x| \leq 2R\}} f \frac{\partial f}{\partial \nu} dl \\
&= f'(r)\, 2\pi r - f'(R)\, 2\pi R \\
&= \frac{1}{(\ln 2)\, r} 2\pi r + \frac{1}{(\ln 2)\, R} 2\pi R \\
&= \frac{4\pi}{\ln 2} < 20.
\end{aligned}$$

Suppose that we have a sequence of annuli $\{A_k\}_{k=1}^N$, with different centers and different radii, but such that the sequence $\{2A_k\}_{k=1}^N$ is disjoint. Then, defining f_k for each pair $(A_k, 2A_k)$ as above, we obtain a sequence of functions with disjoint supports and with

$$\int_{\mathbb{R}^2} |\nabla f_k|^2 \, dx < 20.$$

Note that

$$\int_{\mathbb{R}^2} V f_k^2 dx \geq \int_{A_k} V dx.$$

Hence, the condition $\int_{\mathbb{R}^2} |\nabla f_k|^2 \, dx < \int_{\mathbb{R}^2} V f_k^2 dx$ will be satisfied if

$$\int_{A_k} V dx \geq 20.$$

Consider again measure μ given by $d\mu = V dx$ and restate our problem as follows: construct N annuli $\{A_k\}_{k=1}^N$ such that

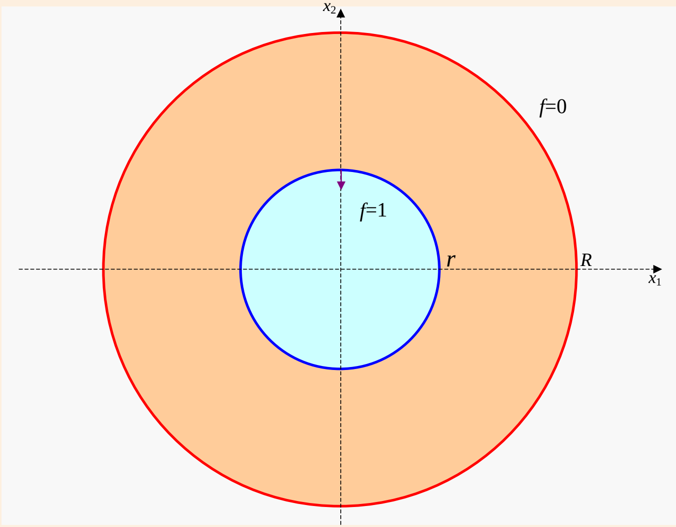
(i) $\{2A_k\}_{k=1}^N$ are disjoint,

(ii) $\mu(A_k) \geq 20$ for each k,

(iii) and $N \geq c\mu(\mathbb{R}^2)$.

Of course, if $\mu(\mathbb{R}^2) < 20$ then such a sequence cannot be constructed. In this case we argue differently. Choose some $0 < r < R$ and consider the function

$$f(x) = \begin{cases} 1, & |x| \le r \\ 0, & x \ge R, \\ \frac{1}{\ln \frac{R}{r}} \ln \frac{R}{|x|}, & r \le |x| \le R. \end{cases}$$



For this function

$$\int_{\mathbb{R}^2} |\nabla f|^2 \, dx = -f'(r) \, 2\pi r = \frac{2\pi}{\ln \frac{R}{r}}$$

while

$$\int_{\mathbb{R}^2} V f^2 dx \geq \int_{\{|x| \leq r\}} V dx.$$

Taking r and $\frac{R}{r}$ large enough, we obtain $\int_{\mathbb{R}^2} |\nabla f|^2 \, dx < \int_{\mathbb{R}^2} V f^2 dx$ whence $\mathcal{N}_0 (H_V) \geq 1$. If $\mu(\mathbb{R}^2) = \int_{\mathbb{R}^2} V dx$ is bounded by some constant, say 20, then we obtain $\mathcal{N}_0 (H_V) \geq c\mu(\mathbb{R}^2)$ just by taking c small enough.

Hence, in the main part we can assume that $\mu(\mathbb{R}^2)$ is large enough. In this case, the sequence of annuli satisfying (i)-(iii) can be always constructed. In fact, the positive answer is given by the following abstract theorem.

Theorem 2.2 *Let (X, d) be a metric space and μ is a non-atomic Borel measure on X. Assume that the following properties are satisfied.*

1. *All metric balls $B(x, r) = \{y \in X : d(x, y) < r\}$ are precompact.*

2. *There exists a constant M such that, for any ball $B(x, r)$ there is a family of at most M balls of radii $r/2$ that cover $B(x, r)$.*

Then there is a constant $c = c(M) > 0$ such that, for any $0 < v < \mu(X)$ there exists at least $c\frac{\mu(X)}{v}$ annuli $\{A_k\}$ such that

(i) *$\{2A_k\}$ are disjoint*

(ii) *and $\mu(A_k) \geq v$ for any k.*

Of course, $\mathbb{R}^2$ satisfies all the hypotheses of Theorem 2.2. Taking $v = 20$ we obtain that if $\mu(\mathbb{R}^2) > 20$ then there exists at least $c'\mu(\mathbb{R}^2)$ annuli satisfying (i) and (ii), which finishes the proof of Theorem 2.1.

We leave Theorem 2.2 without proof, only mentioning that it can be regarded as a sophisticated version of the ball covering argument. Note also that annuli in the statement cannot be replaced by balls.

2.3 Estimates of eigenvalues on $\mathbb{S}^2$

Let us show one more application of Theorem 2.2.

Theorem 2.3 *Let λ_k, $k = 1, 2, \ldots$, be the k-th smallest eigenvalue of the Laplace-Beltrami operator Δ on $(\mathbb{S}^2, g)$, where g is an arbitrary Riemannian metric on $\mathbb{S}^2$. Then, for any k,*

$$\lambda_k \leq C \frac{k-1}{\mu(\mathbb{S}^2)}, \tag{2.2}$$

where C is a universal constant and μ is the Riemannian volume of the metric g.

In fact, this theorem holds also for any closed Riemann surface, where the constant C depends also on the genus of the surface. However, the general case follows from the estimate for $\mathbb{S}^2$.

Note that $\lambda_1 = 0$ so that the case $k = 1$ is trivial. For $k = 2$ Theorem 2.3 was proved by Hersch in 1970 for the sphere and then for any Riemann surface by Yang and Yau in 1980. For a general k, Yau stated (2.2) as a conjecture, which was proved by Korevaar in 1993.

The main point of (2.2) that the constant C does not depend on the Riemannian metric g. The metric enters (2.2) only through the total area $\mu\left(\mathbb{S}^2\right)$. This is essentially two-dimensional phenomenon as such estimates do not hold in higher dimensions.

Let us show how Theorem 2.3 can be obtained from Theorem 2.2. Consider the counting function for Δ on $(\mathbb{S}^2, g)$:

$$\mathcal{N}_\lambda = \#\left\{j \geq 1 : \lambda_j < \lambda\right\}.$$

Note that $\lambda_k < \lambda$ will follow from $\mathcal{N}_\lambda \geq k$. We will prove that, for all $\lambda > 0$,

$$\mathcal{N}_\lambda \geq C^{-1} \mu\left(\mathbb{S}^2\right) \lambda. \tag{2.3}$$

If (2.3) is already proved, then choosing here $\lambda = C\frac{k}{\mu(\mathbb{S}^2)}$, where $k \geq 2$, we obtain $\mathcal{N}_\lambda \geq k$ and, hence,

$$\lambda_k < \lambda = C\frac{k}{\mu\left(\mathbb{S}^2\right)} \leq 2C\frac{k-1}{\mu\left(\mathbb{S}^2\right)},$$

which proves (2.2).

Let us prove (2.3) for any $\lambda > 0$. The counting function admits variational characterization

$$\mathcal{N}_\lambda = \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec D_{\mathbb{S}^2},\ \mathcal{E}(f) < \lambda \|f\|_2^2 \ \forall f \in \mathcal{V} \setminus \{0\}\right\}$$

where

$$\mathcal{E}(f) = \int_{\mathbb{S}^2} |\nabla f|_g^2 \, d\mu \quad \text{and} \quad \|f\|_2^2 = \int_{\mathbb{S}^2} f^2 d\mu.$$

Hence, it suffices to construct at least $N = C^{-1}\mu(\mathbb{S}^2)\lambda$ functions f with disjoint supports and with $\mathcal{E}(f) < \lambda \|f\|_2^2$.

If λ is small enough, namely, if $C^{-1}\mu(\mathbb{R}^2)\lambda \leq 1$ then we need to construct only one function, and it always exists: $f \equiv 1$. Hence, we can assume that $\lambda > \frac{C}{\mu(\mathbb{S}^2)}$.

Any metric g on $\mathbb{S}^2$ is conformally equivalent to the canonical metric g_0 on $\mathbb{S}^2$. Denote by μ_0 the canonical Riemannian measure on $\mathbb{S}^2$. Note that the energy is a conformal invariant:

$$\mathcal{E}(f) = \int_{\mathbb{S}^2} |\nabla f|_g^2 \, d\mu = \int_{\mathbb{S}^2} |\nabla f|_{g_0}^2 \, d\mu_0.$$

Let d be the geodesic distance on $(\mathbb{S}^2, g_0)$. As in $\mathbb{R}^2$, one can show that, for any annulus A on $\mathbb{S}^2$ (with respect to d) one can construct a test function f supported in $2A$ and such that $f|_A = 1$ and $\mathcal{E}(f) < K$ where K is some constant. On the other hand,

$$\|f\|_2^2 \geq \int_A f^2 d\mu = \mu(A),$$

so that $\mathcal{E}(f) < \lambda \|f\|_2^2$ will follow from $K \leq \lambda\mu(A)$. Hence, we need to construct at least $N = C^{-1}\mu(\mathbb{S}^2)\lambda$ annuli A_k on $\mathbb{S}^2$ so that $2A_k$ are disjoint and

$$\mu(A_k) \geq \frac{K}{\lambda}.$$

Let us emphasize that measure μ is defined by the metric g, whereas the annuli are defined using the distance function of g_0.

Let us apply Theorem 2.2 to the metric space $(\mathbb{S}^2, d)$ with measure μ. Set $v := \frac{K}{\lambda} < C^{-1}K\mu(\mathbb{S}^2)$. Choosing $C > K$, we have $v < \mu(\mathbb{S}^2)$ so that Theorem 2.2 can be applied. Hence, we obtain at least $c\frac{\mu(\mathbb{S}^2)}{v} = \frac{c}{K}\mu(\mathbb{S}^2)\lambda$ annuli A_k with disjoint $2A_k$ and with

$$\mu(A_k) \geq v = \frac{K}{\lambda},$$

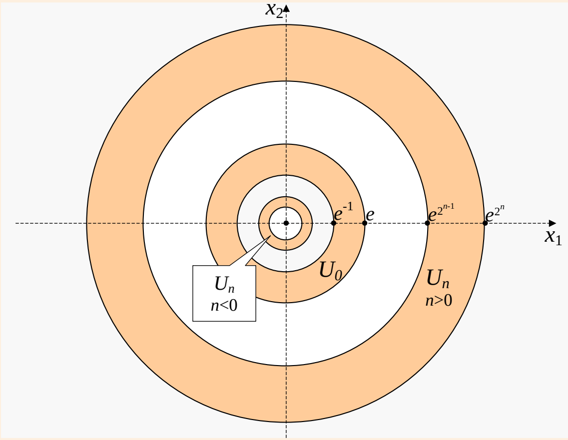
which finishes the proof of (2.3) with $C = \frac{K}{c}$.

3 Upper estimate in $\mathbb{R}^2$

3.1 Statement of the result

Consider a tiling of $\mathbb{R}^2$ into a sequence of annuli $\{U_n\}_{n\in\mathbb{Z}}$ defined by

$$U_n \overset{n\le 0}{=} \{e^{-2^{|n|}} < |x| < e^{-2^{|n|-1}}\}, \quad U_0 = \{e^{-1} < |x| < e\}, \quad U_n \overset{n\ge 0}{=} \{e^{2^{n-1}} < |x| < e^{2^n}\}$$



Given a potential (=a non-negative L^1_{loc}-function) $V(x)$ on $\mathbb{R}^2$ and $p > 1$, define for any $n \in \mathbb{Z}$ the following quantities:

$$A_n = \int_{U_n} V(x)(1 + |\ln|x||)\,dx\,, \quad B_n = \left(\int_{\{e^n<|x|<e^{n+1}\}} V^p(x)\,|x|^{2(p-1)}\,dx \right)^{1/p} \tag{3.1}$$

The main result of this section is the following theorem.

Theorem 3.1 (AG, N.Nadirashvili, 2012) *For any potential V in $\mathbb{R}^2$ and for any $p > 1$, we have*

$$\operatorname{Neg}(V) \le 1 + C \sum_{\{n\in\mathbb{Z}:A_n>c\}} \sqrt{A_n} + C \sum_{\{n\in\mathbb{Z}:B_n>c\}} B_n, \tag{3.2}$$

where C, c are positive constants depending only on p.

The additive term 1 in (3.2) reflects a special feature of $\mathbb{R}^2$: for any non-zero potential V, there is at least 1 negative eigenvalue of H_V, no matter how small are the sums in (3.2), as it was shown in the course of the proof of Theorem 2.1.

Let us compare (3.2) with previously known upper bounds. A simpler (and coarser) version of (3.2) is

$$\operatorname{Neg}(V) \leq 1 + C \int_{\mathbb{R}^2} V(x)(1 + |\ln |x||)\, dx + C \sum_{n \in \mathbb{Z}} B_n. \tag{3.3}$$

Indeed, if $A_n > c$ then $\sqrt{A_n} \leq c^{-1/2} A_n$ so that the first sum in (3.2) can be replaced by $\sum_{n\in\mathbb{Z}} A_n$ thus yielding (3.3).

The estimate (3.3) was obtained by Solomyak in 1994. In fact, he proved a better version:

$$\operatorname{Neg}(V) \leq 1 + C \|A\|_{1,\infty} + C \sum_{n \in \mathbb{Z}} B_n, \tag{3.4}$$

where A denotes the whole sequence $\{A_n\}_{n\in\mathbb{Z}}$ and $\|A\|_{1,\infty}$ is the weak l^1-norm (the Lorentz norm) given by

$$\|A\|_{1,\infty} = \sup_{s>0} s\# \{n : A_n > s\}.$$

Clearly, $\|A\|_{1,\infty} \leq \|A\|_1$ so that (3.4) is better than (3.3).

However, (3.4) also follows from (3.2) using an observation that

$$\|A\|_{1,\infty} \leq \sup_{s>0} s^{1/2} \sum_{\{A_n>s\}} \sqrt{A_n} \leq 4\,\|A\|_{1,\infty}\,.$$

In particular, we have

$$\sum_{\{A_n>c\}} \sqrt{A_n} \leq 4c^{-1/2}\,\|A\|_{1,\infty}\,,$$

so that (3.2) implies (3.4). As we will see below, our estimate (3.2) provides for certain potentials strictly better results than (3.4).

In the case when $V(x)$ is a radial function, that is, $V(x) = V(|x|)$, the following estimate was proved by physicists Chadan, Khuri, Martin, Wu in 2003:

$$\mathrm{Neg}\,(V) \leq 1 + \int_{\mathbb{R}^2} V(x)\,(1 + |\ln |x||)\,dx. \tag{3.5}$$

Although this estimate is better than (3.3), we will see that our main estimate (3.2) gives for certain potentials strictly better estimates than (3.5).

Another upper estimate for a general potential in $\mathbb{R}^2$ was obtained by Molchanov and Vainberg in 2010:

$$\operatorname{Neg}(V) \leq 1 + C \int_{\mathbb{R}^2} V(x) \ln \langle x \rangle \, dx + C \int_{\mathbb{R}^2} V(x) \ln \left(2 + V(x) \langle x \rangle^2\right) dx, \tag{3.6}$$

where $\langle x \rangle = e + |x|$. However, due to the logarithmic term in the second integral, this estimate never implies the linear semi-classical asymptotic

$$\operatorname{Neg}(\alpha V) \simeq O(\alpha) \quad \text{as } \alpha \to \infty, \tag{3.7}$$

that is expected to be true for "nice" potentials. Observe that the Solomyak estimates (3.3) and (3.4) are linear in V so that they imply (3.7) whenever the right hand side is finite.

Our estimate (3.2) gives both linear asymptotic (3.7) for "nice" potentials and non-linear asymptotics for some other potentials. Let us emphasize two main novelties in our estimate (3.2): using the square root of A_n instead of linear expressions, and the restriction of the both sums in (3.2) to the values $A_n > c$ and $B_n > c$, respectively, which allows to obtain significantly better results.

The reason for the terms $\sqrt{A_n}$ in (3.2) can be explained as follows. Different parts of the potential V contributes differently to $\operatorname{Neg}(V)$. The high values of V that are concentrated on relatively small areas, contribute to $\operatorname{Neg}(V)$ via the terms B_n, while the low values of V scattered over large areas, contribute via the terms A_n. Since we integrate V over annuli, the long range effect of V becomes similar to that of an one-dimensional potential. In $\mathbb{R}^1$ one expects $\operatorname{Neg}(\alpha V) \simeq \sqrt{\alpha}$ as $\alpha \to \infty$ which explains the appearance of the square root in (3.2).

By the way, the following estimate of $\operatorname{Neg}(V)$ in $\mathbb{R}^1_+$ was proved by Solomyak:

$$\operatorname{Neg}(V) \leq 1 + C \sum_{n=0}^{\infty} \sqrt{a_n} \tag{3.8}$$

where

$$a_n = \int_{I_n} V(x)(1+|x|)\,dx$$

and $I_n = [2^{n-1}, 2^n]$ if $n > 0$ and $I_0 = [0,1]$. Clearly, the sum $\sum \sqrt{a_n}$ here resembles $\sum \sqrt{A_n}$ in (3.2), which is not a coincidence. In fact, our method allows to improve (3.8) by restricting the sum to those n for which $a_n > c$.

Returning to (3.3), one can apply a suitable Hölder inequality to combine the both terms of (3.3) in one as follows. Assume that $\mathcal{W}(r)$ is a positive monotone increasing function on $(0,+\infty)$ that satisfies the following Dini type condition both at 0 and at ∞:

$$\int_0^\infty \frac{r\,|\ln r|^{\frac{p}{p-1}}\,dr}{\mathcal{W}(r)^{\frac{1}{p-1}}} < \infty. \tag{3.9}$$

Then

$$\operatorname{Neg}(V) \leq 1 + C\left(\int_{\mathbb{R}^2} V^p(x)\,\mathcal{W}(|x|)\,dx\right)^{1/p}, \tag{3.10}$$

where the constant C depends on p and $\mathcal{W}$. Here is an example of a weight function $\mathcal{W}(r)$ that satisfies (3.9):

$$\mathcal{W}(r) = r^{2(p-1)}\langle \ln r\rangle^{2p-1}\ln^{p-1+\varepsilon}\langle \ln r\rangle, \tag{3.11}$$

where $\varepsilon > 0$. In particular, for $p = 2$, we obtain the following estimate:

$$\operatorname{Neg}(V) \leq 1 + C\left(\int_{\mathbb{R}^2} V^2(x)\,|x|^2\,\langle \ln |x|\rangle^3 \ln^{1+\varepsilon}\langle \ln |x|\rangle dx\right)^{1/2}. \tag{3.12}$$

3.2 Examples

Example 1. Assume that, for all $x \in \mathbb{R}^2$,

$$V(x) \leq \frac{\alpha}{|x|^2}$$

for a small enough positive constant α. Then, for all $n \in \mathbb{Z}$,

$$B_n \leq \alpha \left(\int_{\{e^n < |x| < e^{n+1}\}} \frac{1}{|x|^2} dx \right)^{1/p} \simeq \alpha$$

so that $B_n < c$ and the last sum in (3.2) is void, whence we obtain

$$\operatorname{Neg}(V) \leq 1 + C \int_{\mathbb{R}^2} V(x) \left(1 + |\ln |x||\right) dx. \tag{3.13}$$

The estimate (3.13) in this case follows also from the estimate (3.6) of Molchanov and Vainberg.

Example 2. Assume that a potential V satisfies the following condition: for some constant K and all $n \in \mathbb{Z}$,

$$\sup_{\{e^n<|x|<e^{n+1}\}} V \leq K \inf_{\{e^n<|x|<e^{n+1}\}} V. \tag{3.14}$$

For such potential we have

$$B_n \simeq \int_{\{e^n<|x|<e^{n+1}\}} V dx, \tag{3.15}$$

so that (3.3) implies

$$\operatorname{Neg}(V) \leq 1 + C \int_{\mathbb{R}^2} V(x)(1+|\ln|x||)\,dx + C' \int_{\mathbb{R}^2} V(x)\,dx,$$

where the constant C' depends also on K. Of course, the second term here can be absorbed by the first one thus yielding (3.13) with $C = C(K)$.

The estimate (3.13) in this case can be obtained from the estimate (3.5) of Chadan, Khuri, Martin, Wu by comparing V with a radial potential.

Example 3. Let

$$V(x) = \frac{\alpha}{|x|^2 \left(1 + \ln^2 |x|\right)},$$

where $\alpha > 0$ is small enough. Then as in the first example $B_n < c$, while A_n can be computed as follows: for $n > 0$

$$A_n \simeq \int_{e^{2^{n-1}}}^{e^{2^n}} \frac{\alpha}{r^2 \ln^2 r} (\ln r)\, r dr = \alpha \int_{e^{2^{n-1}}}^{e^{2^n}} d \ln \ln r \simeq \alpha, \tag{3.16}$$

and the same for $n \leq 0$, so that $A_n < c$ for all n. Hence, the both sums in (3.2) are void, and we obtain

$$\mathrm{Neg}\,(V) = 1.$$

This result cannot be obtained by any of the previously known estimates. Indeed, in the estimates of Chadan, Khuri, Martin, Wu and of Molchanov, Vainberg one has $\int_{\mathbb{R}^2} V(x)\,(1 + |\ln |x||)\, dx = \infty$, and in the estimate (3.4) of Solomyak one has $\|A\|_{1,\infty} = \infty$. As will be shown below, if $\alpha > 1/4$ then $\mathrm{Neg}\,(V)$ can be ∞. Hence, $\mathrm{Neg}\,(V)$ exhibits a non-linear behavior with respect to the parameter α, which cannot be captured by linear estimates.

Example 4. Assume that $V(x)$ is locally bounded and

$$V(x) = o\left(\frac{1}{|x|^2 \ln^2 |x|}\right) \quad \text{as } x \to \infty.$$

Similarly to the above computation we see that $A_n \to 0$ and $B_n \to 0$ as $n \to \infty$, which implies that the both sums in (3.2) are finite and, hence,

$$\operatorname{Neg}(V) < \infty.$$

This result is also new. Note that in this case the integral $\int_{\mathbb{R}^2} V(x)(1 + |\ln |x||)\, dx$ may be divergent; moreover, the norm $\|A\|_{1,\infty}$ can also be ∞ as one can see in the next example.

Example 5. Choose $q > 0$ and set

$$V(x) = \frac{1}{|x|^2 \ln^2 |x| (\ln \ln |x|)^q} \quad \text{for } |x| > e^2 \tag{3.17}$$

and $V(x) = 0$ for $|x| \leq e^2$. For $n \geq 2$ we have

$$A_n \simeq \int_{e^{2^{n-1}}}^{e^{2^n}} \frac{1}{r^2 \ln^2 r (\ln \ln r)^q} (\ln r)\, r dr = \int_{e^{2^{n-1}}}^{e^{2^n}} \frac{d \ln \ln r}{(\ln \ln r)^q} \simeq \frac{1}{n^q},$$

and, by (3.15),

$$B_n \simeq \int_{e^n}^{e^{n+1}} \frac{1}{r^2 \ln^2 r \left(\ln\ln r\right)^q} r dr = \int_{e^n}^{e^{n+1}} \frac{d \ln r}{\ln^2 r \left(\ln\ln r\right)^q} \simeq \frac{1}{n^2 \ln^q n}.$$

Let α be a large real parameter. Then

$$A_n\left(\alpha V\right) \simeq \frac{\alpha}{n^q}, \tag{3.18}$$

and the condition $A_n\left(\alpha V\right) > c$ is satisfied for $n \leq C\alpha^{1/q}$, whence we obtain

$$\sum_{\{A_n(\alpha V)>c\}} \sqrt{A_n\left(\alpha V\right)} \leq C \sum_{n=1}^{\lceil C\alpha^{1/q}\rceil} \sqrt{\frac{\alpha}{n^q}} \simeq C\sqrt{\alpha}\left(\alpha^{1/q}\right)^{1-q/2} = C\alpha^{1/q}.$$

It is clear that $\sum_n B_n\left(\alpha V\right) \simeq \alpha$. Hence, we obtain from (3.2)

$$\operatorname{Neg}\left(\alpha V\right) \leq C\left(\alpha^{1/q} + \alpha\right). \tag{3.19}$$

If $q \geq 1$ then the leading term here is α. Combining this with (2.1), we obtain

$$\operatorname{Neg}\left(\alpha V\right) \simeq \alpha \text{ as } \alpha \to \infty.$$

If $q > 1$ then this follows also from (3.5) and (3.4); if $q = 1$ then only the estimate (3.4) of Solomyak gives the same result as in this case $A_n \simeq \frac{1}{n}$ and $\|A\|_{1,\infty} < \infty$.

If $q < 1$ then the leading term in (3.19) is $\alpha^{1/q}$ so that

$$\operatorname{Neg}(\alpha V) \leq C\alpha^{1/q}.$$

As was shown by Birman and Laptev, in this case, indeed, $\operatorname{Neg}(\alpha V) \simeq \alpha^{1/q}$ as $\alpha \to \infty$. Observe that in this case $\|A\|_{1,\infty} = \infty$, and neither of the estimates previous estimates (3.3), (3.5), (3.4), (3.6) yields even the finiteness of $\operatorname{Neg}(\alpha V)$, leaving alone the correct rate of growth in α.

Example 6. Let V be a potential in $\mathbb{R}^2$ such that

$$\sum_{n\in\mathbb{Z}} \sqrt{A_n(V)} + \sum_{n\in\mathbb{Z}} B_n(V) < \infty. \tag{3.20}$$

Applying (3.2) to αV, we obtain

$$\operatorname{Neg}(\alpha V) \leq 1 + C\alpha^{1/2} \sum_{n\in\mathbb{Z}} \sqrt{A_n(V)} + \alpha \sum_{n\in\mathbb{Z}} B_n(V).$$

Combining with the lower bound (2.1) and letting $\alpha \to \infty$, we see that

$$c\alpha \int_{\mathbb{R}^2} V\,dx \le \operatorname{Neg}(\alpha V) \le \alpha \sum_{n\in\mathbb{Z}} B_n(V) + o(\alpha),$$

in particular,

$$\operatorname{Neg}(\alpha V) \simeq \alpha \quad \text{as } \alpha \to \infty.$$

Furthermore, if V satisfies the condition (3.14) then, using (3.15), we obtain a more precise estimate

$$\operatorname{Neg}(\alpha V) \simeq \alpha \int_{\mathbb{R}^2} V(x)\,dx \quad \text{as } \alpha \to \infty. \tag{3.21}$$

For example, (3.20) is satisfied for the potential (3.17) of Example 5 with $q > 2$, as it follows from (3.18). By a more sophisticated argument, one can show that (3.21) holds also for $q > 1$.

Example 7. Set $R = e^{2^m}$ where m is a large integer, choose $\alpha > \frac{1}{4}$ and consider the following potential on $\mathbb{R}^2$

$$V(x) = \frac{\alpha}{|x|^2 \ln^2 |x|} \quad \text{if } e < |x| < R$$

and $V(x) = 0$ otherwise. Computing A_n as in (3.16) we obtain $A_n \simeq \alpha$ for any $1 \le n \le m$, whence it follows that

$$\sum_{n\in\mathbb{Z}} \sqrt{A_n} = \sum_{n=1}^{m} \sqrt{A_n} \simeq \sqrt{\alpha} m \simeq \sqrt{\alpha} \ln\ln R.$$

Also, we obtain by (3.15) $B_n \simeq \frac{a}{n^2}$, for $1 \le n < 2^m$, whence

$$\sum_{n\in\mathbb{Z}} B_n(V) \simeq \sum_{n=1}^{2^m-1} \frac{\alpha}{n^2} \simeq \alpha.$$

By (3.2) we obtain

$$\operatorname{Neg}(V) \le C\sqrt{\alpha} \ln\ln R + C\alpha. \tag{3.22}$$

Observe that both (3.4) and (3.5) give in this case a weaker estimate

$$\operatorname{Neg}(V) \leq C\alpha \ln \ln R.$$

Let us estimate $\operatorname{Neg}(V)$ from below. Considering the function

$$f(x) = \sqrt{\ln |x|} \sin \left(\sqrt{\alpha - \frac{1}{4}} \ln \ln |x| \right)$$

that satisfies in the region $\Omega = \{e < |x| < R\}$ the differential equation $\Delta f + V(x) f = 0$, and counting the number N of rings

$$\Omega_k := \left\{ x \in \mathbb{R}^2 : \pi k < \sqrt{\alpha - \frac{1}{4}} \ln \ln |x| < \pi (k+1) \right\}$$

in Ω, we obtain

$$\operatorname{Neg}(V) \geq N \simeq \sqrt{\alpha} \ln \ln R$$

(assuming that $\alpha >> \frac{1}{4}$). On the other hand, (2.1) yields $\operatorname{Neg}(V) \geq c\alpha$. Combining these two estimates, we obtain the lower bound

$$\operatorname{Neg}(V) \geq c \left(\sqrt{\alpha} \ln \ln R + \alpha \right),$$

that matches the upper bound (3.22).

3.3 The energy form revisited

We consider a somewhat different energy form than in $\mathbb{R}^n$, $n \geq 3$. For any open set $\Omega \subset \mathbb{R}^2$, consider a function space

$$\mathcal{F}_{V,\Omega} = \left\{ f \in L^2_{loc}\left(\overline{\Omega}\right) : \int_\Omega |\nabla f|^2 \, dx < \infty, \ \int_\Omega V f^2 dx < \infty \right\}$$

and the quadratic form on $\mathcal{F}_{V,\Omega}$:

$$\mathcal{E}_{V,\Omega}\left(f\right) = \int_\Omega |\nabla f|^2 \, dx - \int_\Omega V f^2 dx. \tag{3.23}$$

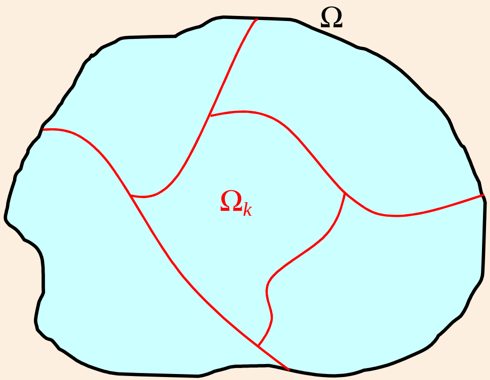
We will use the following quantity:

$$\operatorname{Neg}\left(V,\Omega\right) := \sup\left\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{F}_{V,\Omega} : \mathcal{E}_{V,\Omega}\left(f\right) \leq 0 \text{ for all } f \in \mathcal{V}\right\}. \tag{3.24}$$

Clearly, we have $\mathcal{N}_0\left(H_V\right) \leq \operatorname{Neg}(V, \mathbb{R}^2)$, but in $\mathbb{R}^2$ we do not loose much when we estimate a larger quantity Neg instead of $\mathcal{N}_0$. (Observe that $\mathcal{F}_{V,\mathbb{R}^2}$ contains $f = \text{const}$ and $\mathcal{E}\left(f\right) \leq 0$ so that $\operatorname{Neg}\left(V, \mathbb{R}^2\right) \geq 1$, but as we know, $\mathcal{N}_0\left(H_V\right) \geq 1$). Theorem 3.1 contains the estimate of $\operatorname{Neg}\left(V\right) = \operatorname{Neg}\left(V, \mathbb{R}^2\right)$.

For bounded domains with smooth boundary, $\operatorname{Neg}(V, \Omega)$ is equal to the number of non-positive eigenvalues of the *Neumann* problem in Ω for $-\Delta - V$.

A useful feature of $\operatorname{Neg}(V, \Omega)$ is subadditivity with respect to Ω. We say that a sequence $\{\Omega_k\}$ of open sets $\Omega_k \subset \mathbb{R}^2$ is a *partition* of Ω if all the sets Ω_k are disjoint, $\Omega_k \subset \Omega$, and $\overline{\Omega} \setminus \bigcup_k \Omega_k$ has measure 0.



Lemma 3.2 *If* $\{\Omega_k\}$ *is a partition of* Ω, *then*

$$\operatorname{Neg}(V, \Omega) \le \sum_k \operatorname{Neg}(V, \Omega_k)\,. \tag{3.25}$$

The idea of the proof is the same as in the classical Weyl's argument: adding additional Neumann boundaries inside Ω increases the space of test functions and, hence, the number of non-negative eigenvalues.

3.4 One negative eigenvalue in a disc

Denote by D_r the open disk of radius r in $\mathbb{R}^2$, that is, $D_r = \{x \in \mathbb{R}^2 : |x| < r\}$, and set $D_1 \equiv D$.

Lemma 3.3 *For any $p > 1$ there is $\varepsilon > 0$ such that, for any potential V in D,*

$$\|V\|_{L^p(D)} \leq \varepsilon \Rightarrow \operatorname{Neg}(V, D) = 1.$$

Sketch of proof. Since always $\operatorname{Neg}(V, D) \geq 1$, we need only to prove that $\operatorname{Neg}(V, D) \leq 1$. We will prove that if $u \in \mathcal{F}_{V,D}$ then

$$u \perp 1 \text{ in } L^2(D) \quad \text{and} \quad \mathcal{E}_{V,D}(u) \leq 0 \quad \Rightarrow \quad u = 0,$$

which will imply that $\operatorname{Neg}(V, D) \leq 1$.

Extend $u \in \mathcal{F}_{V,D}$ to $\mathbb{R}^2$ using the inversion $\Phi(x) = \frac{x}{|x|^2}$: for any $x \notin D$, set $u(x) = u(\Phi(x))$. By conformal invariance of energy, we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx = 2 \int_D |\nabla u|^2 \, dx \leq 2 \int_D V u^2 dx.$$

Choose a cutoff function φ such that $\varphi|_{D_2} \equiv 1$, $\varphi|_{\mathbb{R}^2 \setminus D_3} = 0$ and set $u^* = u\varphi$. Then it follows that

$$\int_{D_4} |\nabla u^*|^2 \, dx \leq C \int_D V u^2 dx,$$

with some absolute constant C. Since $u \perp 1$, one uses in the proof the Poincaré inequality in D in the form $\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}$.

Next, we have by Hölder inequality

$$\int_D V u^2 dx \leq \left(\int_D V^p dx \right)^{1/p} \left(\int_D |u|^{\frac{2p}{p-1}} \, dx \right)^{1-1/p},$$

and by Sobolev inequality

$$\left(\int_D |u|^{\frac{2p}{p-1}} \, dx \right)^{1-1/p} \leq \left(\int_{D_4} |u^*|^{\frac{2p}{p-1}} \, dx \right)^{1-1/p} \leq C \int_{D_4} |\nabla u^*|^2 \, dx.$$

Combining the above three lines, we obtain

$$\int_{D_4} |\nabla u^*|^2 \, dx \leq C \left(\int_D V^p dx \right)^{1/p} \int_{D_4} |\nabla u^*|^2 \, dx. \tag{3.26}$$

Assuming that $\|V\|_{L^p(D)}$ is small enough, we see that (3.26) is only possible if $u^* = \text{const}$. Since $u \perp 1$ in $L^2(D)$, it follows that $u \equiv 0$. ∎

Corollary 3.4 *Let Ω be a domain in $\mathbb{R}^2$ that is bilipschitz equivalent to D_r. Then*

$$\int_\Omega V^p dx \le c r^{2-2p} \Rightarrow \operatorname{Neg}(V, \Omega) = 1. \tag{3.27}$$

where $c > 0$ depends on p and on the Lipschitz constant of the mapping between D_r and Ω.

Proof. Indeed, if $\Omega = D_r$ then (3.27) follows from Lemma 3.3 by scaling transformation. For a general Ω one shows that $\operatorname{Neg}(V, \Omega) \le \operatorname{Neg}(CV^*, D_r)$ where V^* is the pull-back of V under the bilipschitz mapping $L : D_r \to \Omega$ where the constant C depends on the Lipschitz constant. ∎

3.5 Negative eigenvalues in a square

Denote by Q the unit square in $\mathbb{R}^2$.

Lemma 3.5 *For any $p > 1$ and for any potential V in Q,*

$$\operatorname{Neg}(V, Q) \leq 1 + C \|V\|_{L^p(Q)}, \tag{3.28}$$

where C depends only on p.

Proof. It suffices to construct a partition $\mathcal{P}$ of Q into a family of N disjoint subsets such that

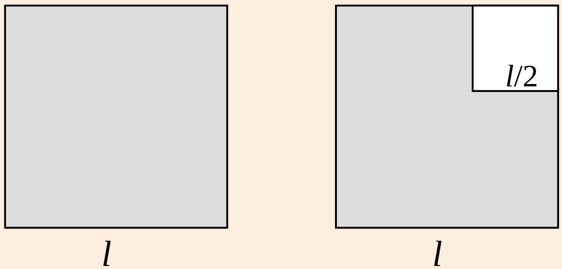
1. $\operatorname{Neg}(V, \Omega) = 1$ for any $\Omega \in \mathcal{P}$;
2. $N \leq 1 + C \|V\|_{L^p(Q)}$.

Indeed, if such a partition exists then we obtain by Lemma 3.2

$$\operatorname{Neg}(V, Q) \leq \sum_{\Omega \in \mathcal{P}} \operatorname{Neg}(V, \Omega) = N, \tag{3.29}$$

and (3.28) follows from the above bound of N.

The elements of a partition will be of two shapes: it is either a square of the side length $0 < l \leq 1$ or a *step*, that is, a set of the form $\Omega = A \setminus B$ where A is a square of the side length l, and B is a square of the side length $\leq l/2$ that is attached to one of corners of A.



In the both cases we refer to l as the size of Ω. By Corollary 3.4, the condition 1 for such a set Ω will follow from

$$\int_\Omega V^p dx \leq c l^{2-2p}. \tag{3.30}$$

Apart from the shape, we will distinguish also the *type* of a set $\Omega \in \mathcal{P}$ of size l as follows: we say that

- Ω is of a large type, if

$$\int_\Omega V^p dx > cl^{2-2p},$$

- Ω is of a medium type if

$$c' l^{2-2p} < \int_\Omega V^p dx \leq cl^{2-2p}, \tag{3.31}$$

- and Ω is of small type if

$$\int_\Omega V^p dx \leq c' l^{2-2p}. \tag{3.32}$$

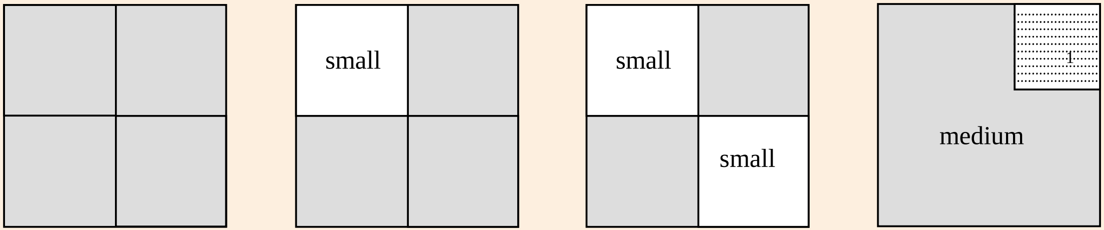
Here c is the constant from (3.30) and $c' \in (0, c)$ will be chosen below. In particular, if Ω is of small or medium type then $\operatorname{Neg}(V, \Omega) = 1$.

The construction of the partition $\mathcal{P}$ will be done by induction. At each step $i \geq 1$ of induction we will have a partition $\mathcal{P}^{(i)}$ of Q such that

1. each $\Omega \in \mathcal{P}^{(i)}$ is either a square or a step;

2. If $\Omega \in \mathcal{P}^{(i)}$ is a step then Ω is of a medium type.

At step 1 we have just one set: $\mathcal{P}^{(1)} = \{Q\}$. At any step $i \geq 1$, partition $\mathcal{P}^{(i+1)}$ is obtained from $\mathcal{P}^{(i)}$ as follows. If $\Omega \in \mathcal{P}^{(i)}$ is small or medium then Ω becomes one of the elements of the partition $\mathcal{P}^{(i+1)}$. If $\Omega \in \mathcal{P}^{(i)}$ is large, then it is a square, and it will be further partitioned into a few sets that will become elements of $\mathcal{P}^{(i+1)}$. Denoting by l the side length of the square Ω, let us first split Ω into four equal squares $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ of side length $l/2$ and consider the following cases.



Case 1. If among $\Omega_1, ..., \Omega_4$ the number of small squares is at most 2, then all sets $\Omega_1, ..., \Omega_4$ become elements of $\mathcal{P}^{(i+1)}$.

Case 2. If among $\Omega_1, ..., \Omega_4$ there are exactly 3 small squares, say, $\Omega_2, \Omega_3, \Omega_4$, then we have

$$\int_{\Omega\setminus\Omega_1} V^p dx = \int_{\Omega_2\cup\Omega_3\cup\Omega_4} V^p dx \leq 3c' \left(\frac{l}{2}\right)^{2-2p} = 3c' 2^{2p-2} l^{2-2p} < cl^{2-2p},$$

where we choose c' to satisfy $3c'2^{2p-2} < c$. On the other hand, we have

$$\int_{\Omega} V^p dx > cl^{2-2p}.$$

Therefore, by reducing the size of Ω_1 (but keeping Ω_1 attached to the corner of Ω) one can achieve the equality

$$\int_{\Omega\setminus\Omega_1} V^p dx = cl^{2-2p}.$$

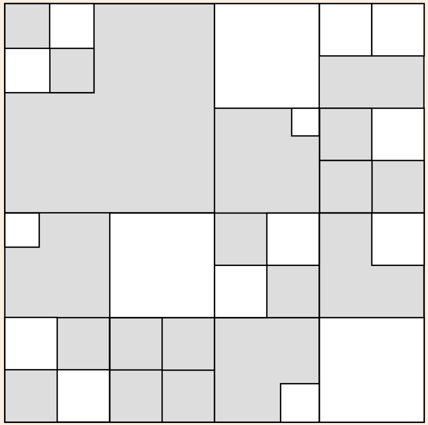
Hence, we obtain a partition of Ω into two sets Ω_1 and $\Omega \setminus \overline{\Omega_1}$, where the step $\Omega \setminus \overline{\Omega_1}$ is of medium type, while the square Ω_1 can be of any type. Both Ω_1 and $\Omega \setminus \overline{\Omega_1}$ become elements of $\mathcal{P}^{(i+1)}$.

Case 3. Let us show that all 4 squares $\Omega_1, ..., \Omega_4$ cannot be small. Indeed, in this case we would have

$$\int_{\Omega} V^p dx = \sum_{k=1}^{4} \int_{\Omega_k} V^p dx \leq 4c' \left(\frac{l}{2}\right)^{2-2p} = \left(4c' 2^{2p-2}\right) l^{2-2p}.$$

Let us choose c' so small that $4c' 2^{2p-2} < c$. Then the above estimate contradicts the assumption that Ω is of large type.

As we see from construction, at each step i only large squares get partitioned further, and the size of the large type squares in $\mathcal{P}^{(i+1)}$ reduces at least by a factor 2. If the size of a square is small enough then it is necessarily of small type, because the right hand side of (3.32) goes to ∞ as $l \to 0$. Hence, the process will stop after finitely many steps. After sufficiently many steps we obtain a partition $\mathcal{P}$ where all the elements are either of small or medium types. In particular, we have $\operatorname{Neg}(V, \Omega) = 1$ for any $\Omega \in \mathcal{P}$.



Let N be a number of elements of $\mathcal{P}$. We need to show that

$$N \leq 1 + C \left\| V \right\|_{L^p(Q)} . \tag{3.33}$$

At each step of construction, denote by L the number of large elements, by M the number of medium elements, and by S the number of small elements. Let us show that the quantity $2L + 3M - S$ is non-decreasing during the construction. Indeed, at each step we split one large square Ω, so that by removing this square, L decreases by 1. However, we add new elements of partitions, which contribute to the quantity $2L+3M-S$ as follows.

1. If Ω is split into $s \leq 2$ small and $4-s$ medium/large squares as in Case 1, then the value of $2L+3M-S$ has the increment at least

$$-2+2(4-s)-s=6-3s\geq 0.$$

2. If Ω is split into 1 square and 1 step as in Case 2, then one obtains at least 1 medium set and at most 1 small, so that $2L+3M-S$ has the increment at least

$$-2+3-1=0.$$

(Luckily, Case 3 cannot occur. In that case, we would have 4 new small squares so that L and M would not have increased, whereas S would have increased at least by 3, so that no quantity of the type C_1L+C_2M-S would have been monotone increasing).

Since for the partition $\mathcal{P}^{(1)}$ we have $2L+3M-S\geq -1$, this inequality will remain true at all steps of construction and, in particular, it is satisfied for the final partition $\mathcal{P}$. For the final partition we have $L = 0$, whence it follows that $S \le 1 + 3M$ and, hence,

$$N = S + M \le 1 + 4M. \tag{3.34}$$

Let us estimate M. Let $\Omega_1, ..., \Omega_M$ be the medium type elements of $\mathcal{P}$ and let l_k be the size of Ω_k. Each Ω_k contains a square $\Omega'_k \subset \Omega_k$ of the size $l_k/2$, and all the squares $\{\Omega'_k\}_{k=1}^M$ are disjoint, which implies that

$$\sum_{k=1}^{M} l_k^2 \le 4. \tag{3.35}$$

Using the Hölder inequality and (3.35), we obtain

$$M = \sum_{k=1}^{M} l_k^{\frac{2}{p'}} l_k^{-\frac{2}{p'}} \le \left(\sum_{k=1}^{M} l_k^2\right)^{1/p'} \left(\sum_{k=1}^{M} l_k^{-\frac{2p}{p'}}\right)^{1/p} \le 4^{1/p'} \left(\sum_{k=1}^{M} l_k^{2-2p}\right)^{1/p}.$$

Since by (3.31) $c' l_k^{2-2p} < \int_{\Omega_k} V^p dx$, it follows that

$$M \le C \left(\sum_{k=1}^{M} \int_{\Omega_k} V^p dx\right)^{1/p} \le C \left(\int_Q V^p dx\right)^{1/p}.$$

Combining this with $N \leq 1 + 4M$, we obtain $N \leq 1 + C \|V\|_{L^p(Q)}$, thus finishing the proof. ■

Corollary 3.6 *Let Ω be a domain in $\mathbb{R}^2$ that is bilipschitz equivalent to D. Then*

$$\mathrm{Neg}\,(V, \Omega) \leq 1 + C \left(\int_\Omega V^p dx \right)^{1/p},$$

where $C > 0$ depends on p and on the Lipschitz constant of the mapping between D and Ω.

3.6 One negative eigenvalue in $\mathbb{R}^2$

Now we would like to obtain conditions for $\operatorname{Neg}(V,\mathbb{R}^2)=1$ in terms of some weighted L^1-norms. The method that we have used in the case $n\geq 3$ (Proposition 1.3) was based on the operator $\mathcal{L}_V=-\frac{1}{V}\Delta$ and estimating of $\left\|\mathcal{L}_V^{-1}\right\|$ in $L^2(\mathbb{R}^n,V\,dx)$.

The hidden reason why it was possible is the existence of the positive Green function $g(x,y)=\frac{c_n}{|x-y|^{n-2}}$ of $-\Delta$. In fact, the operator $\mathcal{L}_V^{-1}$ is given by

$$\mathcal{L}_V^{-1}f=\int_{\mathbb{R}^n}g(x,y)\,f(y)\,V(y)\,dy.$$

The application of the Sobolev in the proof of Proposition 1.3 can be replaced by a direct estimate of the norm of this integral operator in $L^2(\mathbb{R}^n,V\,dx)$. In fact, the classical proof of the Sobolev inequality uses this approach.

One of the difficulties in $\mathbb{R}^2$ is the absence of a positive Green function of the Laplace operator. To overcome this difficulty, we introduce an auxiliary potential $V_0\in C_0^\infty(\mathbb{R}^2)$, such that $V_0\not\equiv 0$ and $V_0\geq 0$.

Lemma 3.7 (AG, 2006) *Operator $H_0 = -\Delta + V_0$ has a positive Green function $g(x,y)$ that admits the following estimate*

$$g(x,y) \simeq \ln\langle x\rangle \wedge \ln\langle y\rangle + \ln_+ \frac{1}{|x-y|}, \tag{3.36}$$

where $\langle x\rangle := e + |x|$ and $\wedge$ means min.

By Lemma 3.3 there exists V_0 such that $\operatorname{Neg}(V_0, \mathbb{R}^2) = 1$. Fix such V_0 and, hence, the Green function $g(x,y)$ of H_0 for what follows.

For a given potential V, define as measure ν by $d\nu = V dx$ and consider the integral operator G_V defined by

$$G_V f(x) = \int_{\mathbb{R}^2} g(x,y) f(y)\, d\nu(y).$$

Denote by $\|G_V\|$ the norm of G_V in the space $L^2(\mathbb{R}^2, \nu)$.

Lemma 3.8 *If* $\|G_V\| \leq \frac{1}{2}$ *then* $\operatorname{Neg}(V, \mathbb{R}^2) = 1.$

Sketch of the proof. The idea is that the operator G_V is the inverse of the operator $\frac{1}{V}H_0$ in $L^2(\nu)$ so that $\|G_V\| \leq \frac{1}{2}$ implies that the spectrum of $\frac{1}{V}H_0$ is confined in $[2, \infty)$. This implies that $H_0 \geq 2V$ in the sense of quadratic forms, that is,

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} V_0 u^2 dx \geq 2 \int_{\mathbb{R}^2} V u^2 dx$$

for all $u \in \mathcal{F}_V$. If $\mathcal{V}$ is a subspace of $\mathcal{F}_V$ where $\mathcal{E}_V \leq 0$ then for any $u \in \mathcal{V}$

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \leq \int_{\mathbb{R}^2} V u^2 dx.$$

Combining the two lines, we obtain

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \leq \int_{\mathbb{R}^2} V_0 u^2 dx,$$

that is, $\mathcal{E}_{V_0}(u) \leq 0$. Taking $\sup \dim \mathcal{V}$ we obtain

$$\operatorname{Neg}\left(V, \mathbb{R}^2\right) \leq \operatorname{Neg}\left(V_0, \mathbb{R}^2\right) = 1.$$

■

The next step is estimating the norm $\|G_V\|$ in terms of V. Since $g(x,y)$ is symmetric in x,y, we have a simple estimate

$$\|G_V\| \le \sup_y \int_{\mathbb{R}^2} g(x,y)\, d\nu(x),$$

which together with Lemma 3.7 leads to

$$\|G_V\| \le C \int_{\mathbb{R}^2} \ln \langle x \rangle \, d\nu(x) + C \sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} \ln_+ \frac{1}{|x-y|} d\nu(x).$$

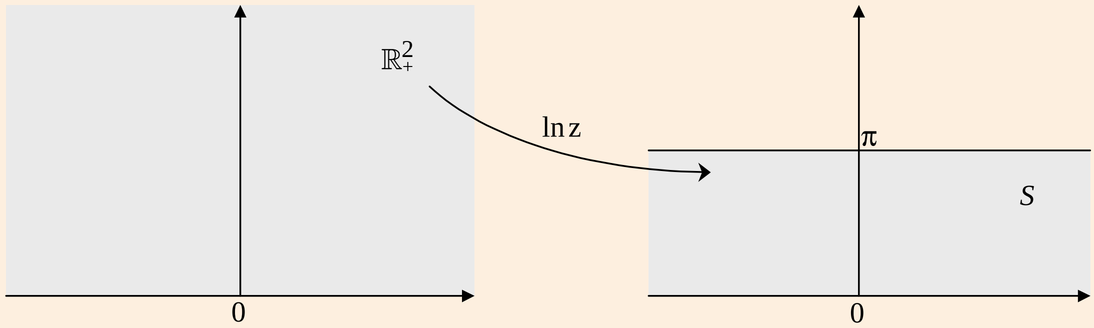
However, $\|G_V\|$ admits a better estimate, as will be explained below.

3.7 Transformation to a strip

It will be more convenient to estimate first $\operatorname{Neg}(V, S)$ where S is a strip in $\mathbb{R}^2$ defined by

$$S = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R},\ 0 < x_2 < \pi\right\}.$$

The strip S is the image of $\mathbb{R}^2_+$ under the conformal mapping $z \mapsto \ln z$:



Let $\gamma(x, y)$ be the push-forward of the Green function $g(x, y)$ under this mapping, that is,

$$\gamma(x, y) = g(e^x, e^y).$$

Using the estimate (3.37) of g, it is possible to show that

$$\gamma\left(x,y\right)\leq C\left\langle x_1\right\rangle\wedge\left\langle y_1\right\rangle+C\ln_+\frac{1}{\left|x-y\right|}. \tag{3.37}$$

For example, x_1 arises from $\ln\left|e^x\right|=\ln\left|e^{x_1+ix_2}\right|=\ln e^{x_1}=x_1$.

Consider also the corresponding integral operator

$$\Gamma_V f\left(x\right)=\int_S\gamma\left(x,y\right)f\left(y\right)d\nu\left(y\right), \tag{3.38}$$

where measure ν is defined as above by $d\nu=V\left(x\right)dx$. Denote by $\left\|\Gamma_V\right\|$ the norm of Γ_V in $L^2\left(S,\nu\right)$. Lemma 3.8 implies the following.

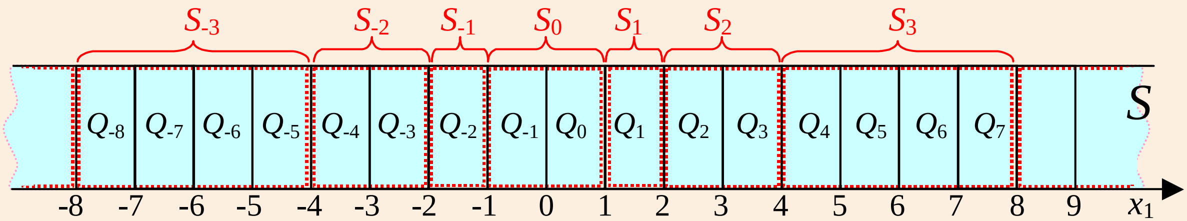
Lemma 3.9 $\left\|\Gamma_V\right\|\leq\frac{1}{8}\Rightarrow\operatorname{Neg}\left(V,S\right)=1$.

The main point in the proof is that the holomorphic mappings are conformal and, hence, preserve the Dirichlet integral.

3.8 Estimating $\|\Gamma_V\|$

For any $n \in \mathbb{Z}$ set

$$Q_n = S \cap \{n < x_1 < n+1\},$$
$$S_n = S \cap \left\{-2^{|n|} < x_1 < -2^{|n|-1}\right\} \text{ for } n < 0,$$
$$S_0 = S \cap \{-1 < x_1 < 1\},$$
$$S_n = S \cap \{2^{n-1} < x_1 < 2^n\} \text{ for } n > 0,$$



$$a_n(V) = \int_{S_n} (1 + |x_1|)\, V(x)\, dx \simeq 2^{|n|} \int_{S_n} V(x)\, dx \qquad (3.39)$$

$$b_n(V) = \left(\int_{Q_n} V^p(x)\, dx \right)^{1/p}. \qquad (3.40)$$

Lemma 3.10 *The operator* Γ_V *admits the following norm estimate in* $L^2(S,\nu)$*:*

$$\|\Gamma_V\| \le C \sup_{n\in\mathbb{Z}} a_n(V) + C \sup_{n\in Z} b_n(V). \qquad (3.41)$$

Approach to the proof. Note that by (3.37)

$$\begin{aligned} |\Gamma_V f(x)| &\le C \int_S (1 + |x_1| \wedge |y_1|) |f(y)| V(y)\, dy \\ &\quad + C \int_S \ln_+ \frac{1}{|x-y|} f(y) |V(y)|\, dy. \end{aligned} \qquad (3.42)$$

The second integral operator can be estimated by the Hölder inequality:

$$\begin{aligned} \int_S \ln_+ \frac{1}{|x-y|} V(y)\, dy &\le \left(\int_{B(x,1)} \left(\ln_+ \frac{1}{|x-y|} \right)^{p'} dy \right)^{1/p'} \\ &\quad \left(\int_{B(x,1)\cap S} V^p(y)\, dy \right)^{1/p}. \end{aligned}$$

The first integral here is equal to a finite constant depending only on p, but independent of x. The second integral is bounded by $C \sup_n b_n(V)$.

It is much more subtle to estimate the norm of the first integral operator in (3.42) via $C\sup_{n\in\mathbb{Z}} a_n(V)$. This problem is reduced to an one dimensional problem by integrating in the direction x_2. Then we apply a certain weighted Hardy inequality. We skip the details as the argument is quite lengthy. ■

Corollary 3.11 *There is a constant $c > 0$ such that*

$$\sup_n a_n(V) \leq c \quad \textit{and} \quad \sup_n b_n(V) \leq c \quad \Rightarrow \quad \operatorname{Neg}(V, S) = 1.$$

Proof. Assuming that the constant c here is small enough, we obtain from (3.41) that $\|\Gamma_V\| \leq \frac{1}{8}$, whence by Lemma 3.9 $\operatorname{Neg}(V, S) = 1$. ■

3.9 Rectangles

For all $\alpha \in [-\infty, +\infty)$, $\beta \in (-\infty, +\infty]$ such that $\alpha < \beta$, denote by $P_{\alpha,\beta}$ the rectangle

$$P_{\alpha,\beta} = \left\{(x_1, x_2) \in \mathbb{R}^2 : \alpha < x_1 < \beta, \quad 0 < x_2 < \pi\right\}.$$

Note that $P_{\alpha,\beta} \subset S$.

Lemma 3.12 *For any potential V in a rectangle $P_{\alpha,\beta}$ with the length $\beta - \alpha \geq 1$, we have*

$$\operatorname{Neg}\left(V, P_{\alpha,\beta}\right) \leq \operatorname{Neg}\left(17V, S\right),$$

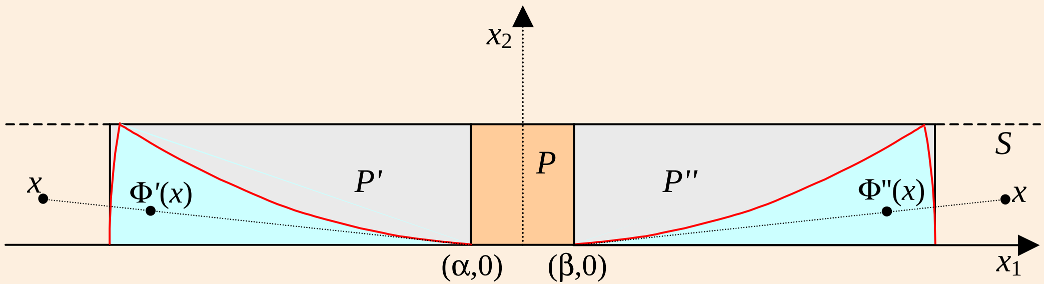
where V is extended to S by setting $V = 0$ outside $P_{\alpha,\beta}$.

Sketch of the proof. It suffices to show that any function $u \in \mathcal{F}_{V,P}$ can be extended to $\mathcal{F}_{V,S}$ so that

$$\int_S |\nabla u|^2 \, dx \leq 17 \int_P |\nabla u|^2 \, dx. \tag{3.43}$$

Attach to P from each side one rectangle, say P' from the left and P'' from the right, each having the length $4(\beta - \alpha)$ (to ensure that the latter is $> \pi$). Extend function u to P' by applying four times symmetries in the vertical sides, so that

$$\int_{P'} |\nabla u|^2 \, dx = 4 \int_P |\nabla u|^2 \, dx.$$



Then slightly reduce P' by taking intersections with the circle of radii $\beta - \alpha$ centered at $(\alpha, 0)$. Now we extend u from P' to the left by using the inversion Φ' at the point $(\alpha, 0)$ in the aforementioned circle. By the conformal invariance of the Dirichlet integral, we have

$$\int_{S\cap\{x_1<\alpha\}} |\nabla u|^2 \leq 8 \int_P |\nabla u|^2 \, dx.$$

Extending u in the same way to the right of P, we obtain (3.43). ■

3.10 Sparse potentials

We say that a potential V in S is *sparse* if

$$\sup_n b_n (V) < c_0,$$

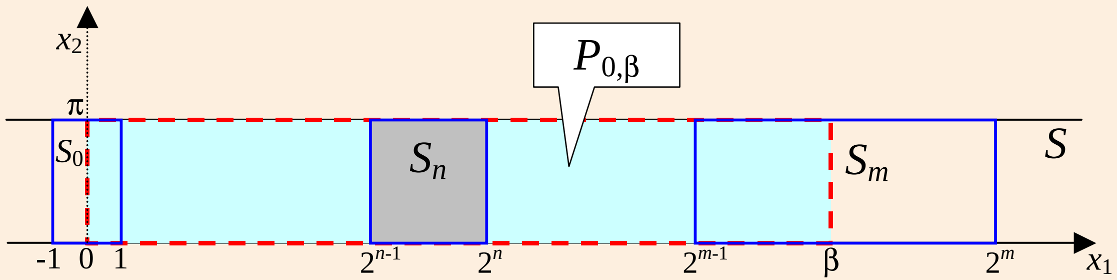
where c_0 is a small enough positive constant, depending only on p. It follows from Corollary 3.11 that, for a sparse potential,

$$\sup_n a_n (V) \leq c \;\Rightarrow \operatorname{Neg}(V, S) = 1.$$

Corollary 3.13 *Let V be a sparse potential in $P_{\alpha,\beta}$ where $\beta - \alpha \geq 1$. Then*

$$(\beta - \alpha) \int_{P_{\alpha,\beta}} V(x)\, dx \leq c \quad \Rightarrow \quad \mathrm{Neg}\,(V, P_{\alpha,\beta}) = 1. \tag{3.44}$$

Proof. Take $\alpha = 0$ so that $\beta \geq 1$. Let m be a non-negative integer such that $2^{m-1} < \beta \leq 2^m$.



Then $a_n(V) = 0$ for $n < 0$ and for $n \geq m+1$. For $0 \leq n \leq m$

$$a_n(V) \leq 2^{n+1} \int_{S_n} V(x)\, dx \leq 2^{m+1} \int_{P_{0,\beta}} V(x)\, dx \leq 4\beta \int_{P_{0,\beta}} V(x)\, dx, \tag{3.45}$$

so that $a_n(17V)$ are small enough for all $n \in \mathbb{Z}$. By Corollary 3.11 $\operatorname{Neg}(17V, S) = 1$, and by Lemma 3.12 $\operatorname{Neg}(V, P_{0,\beta}) = 1$. ■

Lemma 3.14 *Let V be a sparse potential in $P_{\alpha,\beta}$ where $\beta - \alpha \geq 1$. Then*

$$\operatorname{Neg}(V, P_{\alpha,\beta}) \leq 1 + C\left((\beta - \alpha)\int_{P_{\alpha,\beta}} V(x)\,dx\right)^{1/2}. \tag{3.46}$$

In particular, for a sparse potential in S_n,

$$\operatorname{Neg}(V, S_n) \leq 1 + C\sqrt{a_n(V)}. \tag{3.47}$$

Proof. Without loss of generality set $\alpha = 0$. Set also

$$J = \int_{P_{0,\beta}} V(x)\,dx$$

and recall that, by Corollary 3.13, if $\beta J \leq c$ for sufficiently small c then $\operatorname{Neg}(V, P_{0,\beta}) = 1$. Hence, in this case (3.46) is trivially satisfied, and we assume in the sequel that $\beta J > c$.

Due to Lemma 3.12, it suffices to prove that

$$\operatorname{Neg}(V,S) \leq C(\beta J)^{1/2}.$$

Consider a sequence of reals $\{r_k\}_{k=0}^{N}$ such that

$$0 = r_0 < r_1 < ... < r_{N-1} < \beta \leq r_N$$

and the corresponding sequence of rectangles

$$R_k := P_{r_{k-1},r_k} = \{(x_1,x_2) : r_{k-1} < x_1 < r_k, \quad 0 < x_2 < \pi\}$$

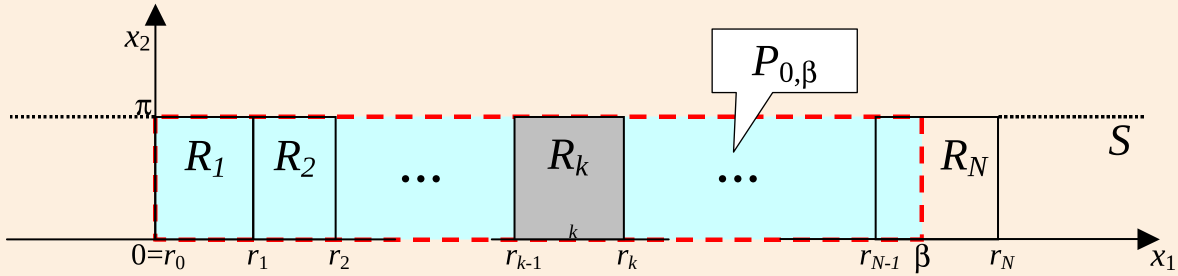
where $k = 1, ..., N$, that covers $P_{0,\beta}$.

Denote $l_k = r_k - r_{k-1}$ and $J_k = \int_{R_k} V(x)\,dx$. By Corollary 3.13,

$$l_k \geq 1 \text{ and } \ l_k J_k \leq c \quad \Rightarrow \ \operatorname{Neg}(V,R_k) = 1 \tag{3.48}$$

Let us construct the sequence $\{r_k\}_{k=0}^{N}$ to satisfy (3.48) for all $k = 1, ..., N$. If r_{k-1} is already defined and $r_{k-1} < \beta$ then choose $r_k > r_{k-1}$ to satisfy the identity

$$l_k J_k = c. \tag{3.49}$$



If such r_k does not exist then set $r_k = \beta+1$; in this case, we have $l_k J_k < c$. Let us show that in the both cases $l_k = r_k - r_{k-1} \geq 1$. Indeed, if $l_k < 1$ then $r_k < \beta + 1$ so that (3.49) is satisfied. By Hölder inequality, (3.49) and $l_k < 1$, we obtain

$$\left(\int_{R_k} V^p dx\right)^{1/p} \geq \frac{1}{(\pi l_k)^{1/p'}} \int_{R_k} V dx = \frac{c}{(\pi l_k)^{1/p'} l_k} \geq \frac{c}{\pi^{1/p'}},$$

which contradicts the assumption that V is sparse. Hence, $l_k \geq 1$.

As soon as we reach $r_k \geq \beta$ we stop the process and set $N = k$. Since always $l_k \geq 1$, the process will indeed stop in a finite number of steps.

We obtain a partition of S into N rectangles $R_1, ..., R_N$ and two half-strips: $S \cap \{x_1 < 0\}$ and $S \cap \{x_1 > r_N\}$, and in the both half-strips we have $V \equiv 0$. In each R_k we have $\operatorname{Neg}(V, R_k) = 1$ whence it follows that

$$\operatorname{Neg}(V, S) \leq 2 + \sum_{k=1}^{N} \operatorname{Neg}(V, R_k) = N + 2.$$

Let us estimate N from above. In each R_k with $k \leq N - 1$ we have by (3.49) $\frac{1}{J_k} = \frac{1}{c} l_k$. Therefore, we have

$$N - 1 = \sum_{k=1}^{N-1} \frac{1}{\sqrt{J_k}} \sqrt{J_k} \leq \left(\frac{1}{c} \sum_{k=1}^{N-1} l_k \right)^{1/2} \left(\sum_{k=1}^{N-1} J_k \right)^{1/2} \leq \left(\tfrac{1}{c} \beta \right)^{1/2} J^{1/2}.$$

Using also $3 \leq 3 \left(\frac{1}{c} \beta J \right)^{1/2}$, we obtain $N + 2 \leq 4 \left(\frac{1}{c} \beta J \right)^{1/2}$, which finishes the proof of (3.46).

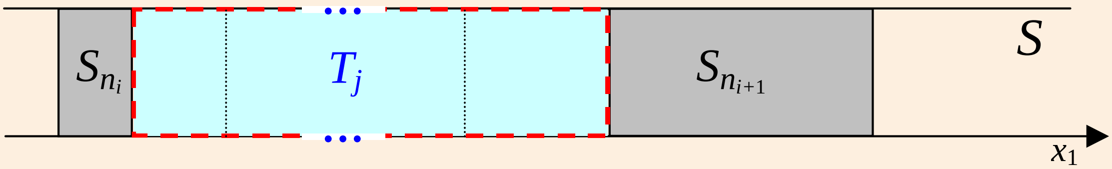
The estimate (3.47) follows trivially from (3.46) and (3.39) as S_n is a rectangle $P_{\alpha,\beta}$ with the length $1 \leq \beta - \alpha \leq 2^{|n|+1}$. $\blacksquare$

Proposition 3.15 *For any sparse potential in the strip* S*,*

$$\operatorname{Neg}(V,S) \leq 1 + C \sum_{\{n: a_n(V) > c\}} \sqrt{a_n(V)}, \tag{3.50}$$

for some constant $C, c > 0$ *depending only on* p*.*

Proof. Let us enumerate in the increasing order those values n where $a_n(V) > c$. So, we obtain an increasing sequence $\{n_i\}$, finite or infinite, such that $a_{n_i}(V) > c$ for any index i. The difference $S \setminus \bigcup_i S_{n_i}$ can be partitions into a sequence $\{T_j\}$ of rectangles, where each rectangle T_j either fills the gap in S between successive rectangles S_{n_i} or T_j may be a half-strip that fills the gap between S_{n_i} and $+\infty$ or $-\infty$.



By construction, each T_j is a union of some rectangles S_k with $a_k(V) \leq c$. It follows from Corollary 3.11 that $\operatorname{Neg}(V, T_j) = 1$. Since by construction

$$\#\{T_j\} \leq 1 + \#\{S_{n_i}\},$$

it follows that

$$\begin{aligned}
\operatorname{Neg}(V, S) &\leq \sum_j \operatorname{Neg}(V, T_i) + \sum_i \operatorname{Neg}(V, S_{n_i}) \\
&\leq 1 + \#\{S_{n_i}\} + \sum_i \operatorname{Neg}(V, S_{n_i}) \\
&\leq 1 + 2\sum_i \operatorname{Neg}(V, S_{n_i}).
\end{aligned}$$

In each S_{n_i} we have by (3.47) and $a_{n_i}(V) > c$ that

$$\operatorname{Neg}(V, S_{n_i}) \leq C\sqrt{a_{n_i}(V)}.$$

Substituting into the previous estimate, we obtain (3.50). ■

3.11 Arbitrary potentials in a strip

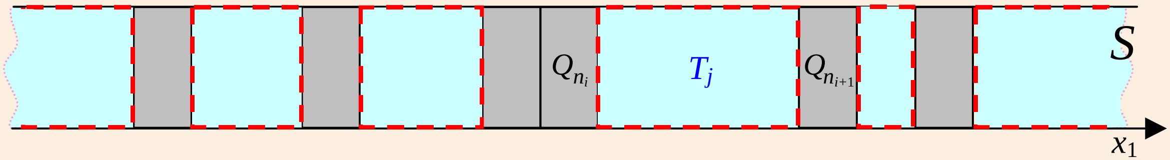
We use notation $a_n(V)$ and $b_n(V)$ defined by (3.39) and (3.40).

Theorem 3.16 *For any $p > 1$ and for any potential V in the strip S, we have*

$$\operatorname{Neg}(V,S) \le 1 + C \sum_{\{n\in\mathbb{Z}: a_n(V) > c\}} \sqrt{a_n(V)} + C \sum_{\{n\in\mathbb{Z}: b_n(V) > c\}} b_n(V), \quad (3.51)$$

where the positive constants C, c depend only on p.

Proof. Let $\{n_i\}$ be a sequence of all $n \in \mathbb{Z}$ for which $b_n(V) > c$. Let $\{T_j\}$ be rectangles that fill the gaps in S between successive Q_{n_i} or between Q_{n_i} and $\pm\infty$.



If the sequence $\{n_i\}$ is empty then V is sparse, and (3.51) follows from Proposition 3.15. Assume that $\{n_i\}$ is non-empty.

Consider the potentials $V' = V\mathbf{1}_{\cup T_j}$ and $V'' = V\mathbf{1}_{\cup Q_{n_i}}$. Since $V = V' + V''$, we have

$$\operatorname{Neg}(V, S) \leq \operatorname{Neg}(2V', S) + \operatorname{Neg}(2V'', S).$$

The potential $2V'$ is sparse by construction, whence by Proposition 3.15

$$\operatorname{Neg}(2V', S) \leq 1 + C \sum_{\{n: a_n(V') > c\}} \sqrt{a_n(V')}. \tag{3.52}$$

By Lemma 3.2 and Lemma 3.5, we obtain

$$\begin{aligned}
\operatorname{Neg}(2V'', S) &\leq \sum_j \operatorname{Neg}(2V'', T_j) + \sum_i \operatorname{Neg}(2V'', Q_{n_i}) \\
&= \#\{T_j\} + \sum_i \left(1 + C \|2V''\|_{L^p(Q_{n_i})}\right) \\
&= \#\{T_j\} + \#\{Q_{n_i}\} + 2C \sum_i b_{n_i}(V).
\end{aligned}$$

By construction we have $\#\{T_j\} \leq 1 + \#\{Q_{n_i}\}$. By the choice of n_i, we have $1 < c^{-1} b_{n_i}(V)$, whence

$$\#\{T_j\} + \#\{Q_{n_i}\} \leq 1 + 2\#\{Q_{n_i}\} \leq 1 + 2c^{-1} \sum_i b_{n_i}(V) \leq 3c^{-1} \sum_i b_{n_i}(V)$$

Combining these estimates together, we obtain

$$\operatorname{Neg}(2V'', S) \leq C' \sum_i b_{n_i}(V) = C' \sum_{\{n: b_n(V) > c\}} b_n(V) \qquad (3.53)$$

Adding up (3.52) and (3.53) yields

$$\operatorname{Neg}(V, S) \leq 1 + C \sum_{\{n: a_n(V') > c\}} \sqrt{a_n(V')} + C \sum_{\{n: b_n(V) > c\}} b_n(V). \qquad (3.54)$$

Since $V' \leq V$, (3.54) implies (3.51), which finishes the proof. ■

Remark. In fact, we have proved a slightly better inequality (3.54) than (3.51).

3.12 Proof of Theorem 3.1

Let us prove the main Theorem 3.1, that is, for any potential V in $\mathbb{R}^2$,

$$\operatorname{Neg}(V) \leq 1 + C \sum_{\{n\in\mathbb{Z}:A_n>c\}} \sqrt{A_n} + C \sum_{\{n\in\mathbb{Z}:B_n>c\}} B_n, \tag{3.55}$$

where

$$A_n(V) = \int_{U_n} V(x)\left(1+|\ln|x||\right)dx, \quad B_n(V) = \left(\int_{W_n} V^p(x)\,|x|^{2(p-1)}\,dx\right)^{1/p},$$

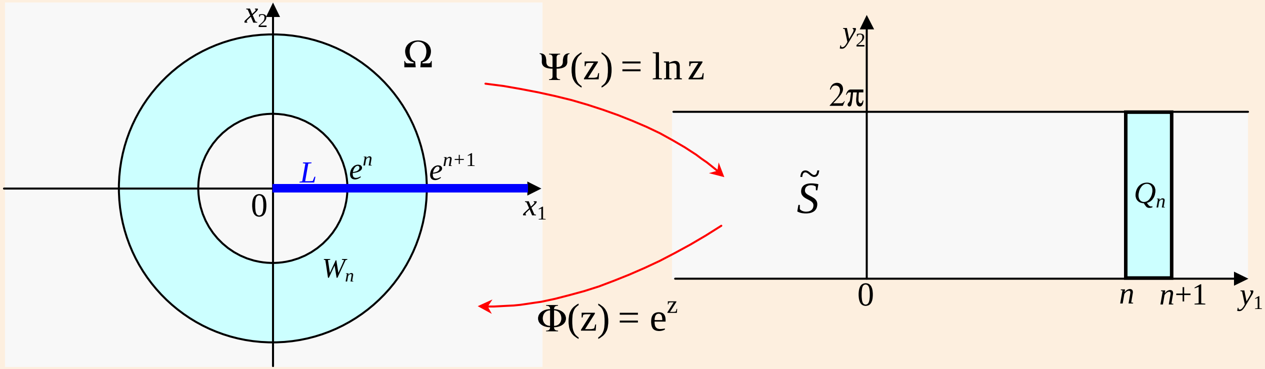
$$U_n = \begin{cases} \{e^{2^{n-1}} < |x| < e^{2^n}\}, & n \geq 1, \\ \{e^{-1} < |x| < e\}, & n = 0, \\ \{e^{-2^{|n|}} < |x| < e^{-2^{|n|-1}}\}, & n \leq -1, \end{cases}$$

and

$$W_n = \left\{e^n < |x| < e^{n+1}\right\}.$$

Consider an open set $\Omega = \mathbb{R}^2 \setminus L$ where $L = \{\, x_1 \geq 0, x_2 = 0\}$ and the mapping $\Psi : \Omega \to \widetilde{S}$ where $\Psi(z) = \ln z$ and

$$\widetilde{S} = \left\{(y_1, y_2) \in \mathbb{R}^2 : \quad 0 < y_2 < 2\pi\right\}.$$



Using the inverse mapping $\Phi = \Psi^{-1}$, define a potential $\widetilde{V}$ on $\widetilde{S}$ by $\widetilde{V}(y) = V(\Phi(y))\,|J_\Phi(y)|$ where J_Φ is the Jacobian of Φ. It is possible to prove that

$$\mathrm{Neg}\left(V, \mathbb{R}^2\right) \leq \mathrm{Neg}\,(V, \Omega) = \mathrm{Neg}(\widetilde{V}, \widetilde{S}). \qquad (3.56)$$

Since the strips $\widetilde{S}$ and S are bilipschitz equivalent, Theorem 3.16 holds also for $\widetilde{S}$, that is,

$$\operatorname{Neg}(\widetilde{V},\widetilde{S}) \leq 1 + C \sum_{\{n: a_n > c\}} \sqrt{a_n} + C \sum_{\{n: b_n(V) > c\}} b_n, \tag{3.57}$$

where

$$a_n = \int_{S_n} (1 + |y_1|)\, \widetilde{V}(y)\, dy, \quad b_n = \left(\int_{Q_n} \widetilde{V}^p dy \right)^{1/p},$$

and

$$Q_n = \Psi(W_n \setminus L), \qquad S_n = \Psi(U_n \setminus L).$$

Since $J_\Psi = \frac{1}{|x|^2}$, we obtain, using the change $y = \Psi(x)$,

$$\begin{aligned} b_n^p &= \int_{Q_n} \widetilde{V}^p(y)\, dy = \int_{W_n} V^p(x)\, |J_\Phi(y)|^p\, |J_\Psi(x)|\, dx \\ &= \int_{W_n} V^p(x)\, |J_\Psi(x)|^{1-p}\, dx \\ &= \int_{W_n} V^p(x)\, |x|^{2(p-1)}\, dx = B_n^p. \end{aligned}$$

Similarly, computing a_n and observing that

$$y_1 = \operatorname{Re}\Psi\left(x\right) = \operatorname{Re}\ln x = \ln\left|x\right|,$$

we obtain

$$\begin{aligned} a_n &= \int_{S_n} \widetilde{V}\left(y\right)\left(1+\left|y_1\right|\right)dy = \int_{U_n} V\left(x\right)\left|J_\Phi\left(y\right)\right|\left(1+\left|\ln\left|x\right|\right|\right)\left|J_\Psi\left(x\right)\right|dx \\ &= \int_{U_n} V\left(x\right)\left(1+\left|\ln\left|x\right|\right|\right)dx = A_n. \end{aligned}$$

Combining together (3.56), (3.57), we obtain (3.55).